GROUP ACTIONS WHOSE SPACE OF INVARIANT MEANS IS FINITE DIMENSIONAL

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Abstract. It is shown that under various set theoretic hypotheses there is an amenable subgroup $\mathbb{F}$ of the full symmetric group on $\mathbb{N}$ such that the space of means on $\mathbb{N}$ invariant under the natural action of $\mathbb{F}$ is finite dimensional, yet the Arens multiplication by invariant means in $\ell^*_\infty(\mathbb{F})$ is never weak* to weak* continuous.

1. Introduction

The purpose of this note is to explore some ramifications of an argument A. T.-M. Lau that can be found in Corollary 5 of [3]. Lau's argument will be used in Proposition 2.2 to show that if a group G acts on a set with a unique invariant mean then Arens multiplication by every invariant mean in $\ell^*_\infty(G)$ is weak* to weak* continuous. (The definitions of these term will be supplied in §2.) The question then arises of whether the assumption of the action of the group having a unique invariant mean can be weakened having the invariant means span a finite dimensional subspace. This will be shown to be consistently false. Since the interest in this question stems from Lau's argument that applies to invariant means in $\ell^*_\infty(\mathbb{G})$, it is natural to only consider the question for groups that have an invariant mean, in other words, only for amenable groups.

2. Key definitions

Definition 2.1. Let $\mathbb{A}$ be a Banach algebra and let $\mathbb{B}$ be a left $\mathbb{A}$-module. Let the module multiplication of $\mathbb{A}$ acting on $\mathbb{B}$ be denoted by $\odot$. Then define maps $\odot_1 : \mathbb{A}^{**} \times \mathbb{B}^{**} \to \mathbb{B}^{**}$ and $\odot_2 : \mathbb{A}^{**} \times \mathbb{B}^{**} \to \mathbb{A}^{**}$ by defining $\odot_1$ and $\odot_2$ with the following equations:

$$\langle \mathfrak{m} \odot_1 \mathfrak{n}, \psi\rangle = \langle \mathfrak{m}, h \mapsto \langle \mathfrak{n}, g \mapsto \psi(h \odot g)\rangle\rangle \tag{2.1}$$

$$\langle \mathfrak{m} \odot_2 \mathfrak{n}, \psi\rangle = \langle \mathfrak{n}, g \mapsto \langle \mathfrak{m}, h \mapsto \psi(h \odot g)\rangle\rangle. \tag{2.2}$$

Note that Equations 2.1 and 2.2 imply that if $h \in \mathbb{A}$ and $g \in \mathbb{B}$ then $h \odot g = h \odot_1 g = h \odot_2 g$. It is also worth observing that the definition does not require that $\mathbb{A}$ be a Banach algebra — all that is needed is that the map $\odot$ be bilinear. This note will focus on the special case of a group $\mathbb{G}$ acting on a set S with the action denoted by $\odot$. This extends naturally to an action of $\ell_1(\mathbb{G})$ on $\ell_1(S)$ and this action will also be denoted by $\odot$. Note that in this context

$$\langle \mathfrak{m} \odot_1 \mathfrak{n}, \psi\rangle = \int\int \psi(gx) d\mathfrak{n}(s) d\mathfrak{m}(g)$$

and that

$$\langle \mathfrak{m} \odot_2 \mathfrak{n}, \psi\rangle = \int\int \psi(gx) d\mathfrak{m}(g) d\mathfrak{n}(s).$$

Notation 2.1. In the following, the fact that each $\mathfrak{n} \in \ell^*_\infty(S)$ can be considered to be a finitely additive measure on S will be used to write $\mathfrak{n}(X)$ instead of the more cumbersome $\mathfrak{n}(1_X)$ or $\langle \mathfrak{n}, 1_X\rangle$ for $X \subseteq S$.

The following has an elementary proof similar to that for the case of Arens products.

Proposition 2.1. *Let G be a group acting on S by $\odot$.*

Research for this paper was partially supported by NSERC of Canada.

(1) *For any* $\mathfrak{n} \in \ell^*_\infty(S)$ *and the mapping* $x \mapsto x \odot_1 \mathfrak{n}$ *is weak*-*weak* continuous on* $\ell^*_\infty(G)$.
(2) *For any* $\mathfrak{m} \in \ell^*_\infty(G)$ *and the mapping* $x \mapsto \mathfrak{m} \odot_2 x$ *is weak*-*weak* continuous on* $\ell^*_\infty(S)$.

Definition 2.2. Define the topological centre $\Lambda_1(\odot)$ of the group action $\odot : \mathbb{G} \times S \to S$ to be the set of $\mathfrak{m} \in \ell^*_\infty(\mathbb{G})$ such that the mapping defined by $x \mapsto \mathfrak{m} \odot_1 x$ is weak* to weak* continuous. Define the topological centre $\Lambda_2(\odot)$ of the group action $\odot_2 : \mathbb{G} \times S \to S$ to be the set of $\mathfrak{n} \in \ell^*_\infty(\mathbb{S})$ such that the mapping defined by $x \mapsto x \odot_2 \mathfrak{n}$ is weak* to weak* continuous.

The following classical result from [4] settles the question of the nature of topological centres for the action of a group on itself.

Theorem 2.1 (A. T.-M. Lau & V. Losert)**.** *If* $(\mathbb{G}, \cdot)$ *is a group acting on itself then no non-trivial positive element of* $\ell^*_\infty(\mathbb{G})$ *is in the topological centre* $\Lambda_1(\cdot)$.

On the other hand, the following argument of Lau yields non-trivial elements of the $\Lambda_1(\odot)$ under some specific circumstances.

Proposition 2.2. *If the following hold:*
(1) $\odot$ *is an action of* $\mathbb{G}$ *on the set* S
(2) $\mathfrak{m} \in \ell^*_\infty(\mathbb{G})$ *is a right invariant mean*
(3) *the action* $\odot$ *has a unique invariant mean in* $\ell^*_\infty(S)$

then $\mathfrak{m} \in \Lambda_1(\odot)$.

Proof. It will be shown that for any $\mathfrak{n} \in \ell^*_\infty(S)$ or norm one $\mathfrak{m} \odot_1 \mathfrak{n}$ is an $\odot$ invariant mean on S. From this and Hypothesis (3) it follows that the mapping $x \mapsto \mathfrak{m} \odot_1 x$ is constant and, hence, continuous. To establish the $\odot$ invariance let $X \subseteq S$ and $h \in \mathbb{G}$. Then

$$\mathfrak{m} \odot_1 \mathfrak{n}(h \odot X) = \int\int 1_{h\odot X}(g \odot x) d\mathfrak{n}(x) d\mathfrak{m}(g) = \mathfrak{m}\left(\{g \in \mathbb{G} \mid g^{-1}h \odot X \in \mathfrak{n}\}\right) =$$

$$\mathfrak{m}\left(\{g \in \mathbb{G} \mid g^{-1} \odot X \in \mathfrak{n}\}\right) = \int\int 1_X(g \odot x) d\mathfrak{n}(x) d\mathfrak{m}(g) = \mathfrak{m} \odot_1 \mathfrak{n}(X)$$

as required. □

In the case of a group acting on itself in the canonical way the only circumstances under which Hypotheses (3) and (2) of Proposition 2.2 can hold is if the group $\mathbb{G}$ is finite. Indeed, in the case of an infinite (discrete) amenable group Lau and Paterson have shown in [5] that the number of invariant means is always $2^{2^{|\mathbb{G}|}}$. However, Yang has shown in [7] under the Continuum Hypothesis — and Foreman under weaker hypotheses in [2] — that there is an amenable subgroup $\mathbb{G}$ of the full symmetric group on $\mathbb{N}$ whose canonical action on $\mathbb{N}$ has a unique invariant mean. Because $\mathbb{G}$ is amenable there is some invariant mean in $\ell^*_\infty(\mathbb{G})$ and it follows from Proposition 2.2 that the topological centre of the action of $\mathbb{G}$ is not empty.

One might ask whether Lau's argument can be improved so as to weaken Hypotheses (3) of Proposition 2.2 asking that only that the invariants means under the action form a finite dimensional subspace. It will be shown in §3 that under the Continuum Hypothesis (as well as various weaker assumptions) there is an amenable subgroup $\mathbb{F}$ of $\mathbb{S}$, the group of all permutations of $\mathbb{N}$, such that the space of means on $\mathbb{N}$ invariant under the action $\odot$ of $\mathbb{F}$ is finite dimensional yet no invariant mean in $\ell^*_\infty(\mathbb{F})$ is in $\Lambda_1(\odot)$.

3. An amenable action of a subgroup of the full symmetric group

Definition 3.1. Given $A \subseteq B \subseteq \mathbb{N}$ an (A, B)-involution is a permutation π of $\mathbb{N}$ such that

- $\pi^2(k) = k$ for all $k \in \mathbb{N}$
- $\pi(k) = k$ for all $k \in \mathbb{N} \setminus B$
- $\pi \restriction (B \setminus A)$ is a bijection from $B \setminus A$ to A.

Notation 3.1. If π is a permutation the support of π is the set of all x such that $\pi(x) \neq x$. $\mathbb{S}_{<\aleph_0}$ will denote the set of all permutations of $\mathbb{N}$ with finite support.

Definition 3.2. Recall that the relation $\subseteq^*$ is defined by $A \subseteq^* B$ if $A \setminus B$ is finite. The cardinal invariant $\mathfrak{p}$ is defined to be the least cardinal such that there is family $\mathcal{F}$ of subsets of $\mathbb{N}$ such that the intersection of any finite subset of $\mathcal{F}$ is infinite yet there is no infinte $Y \subseteq \mathbb{N}$ such that $Y \subseteq^* F$ for all $F \in \mathcal{F}$. An ultrafilter $\mathfrak{u}$ on $\mathbb{N}$ will be called a tower ultrafilter if it is generated by a family $\{A_\xi\}_{\xi\in\mathfrak{p}}$ such that $A_\xi \subseteq^* A_\eta$ if $\xi > \eta$.

The next theorem from [1] will be used in the proof of Lemma 3.1.

Theorem 3.1 (M. Bell). *For any Polish space X without isolated points and any cardinal κ the following are equivalent:*

- $\kappa < \mathfrak{p}$
- *for every family $\mathcal{D}$ of dense open subsets of X the intersection of $\mathcal{D}$ is non-empty provided that $|\mathcal{D}| = \kappa$.*

Lemma 3.1. *If there is a tower ultrafilter and $\mathcal{B}$ is a family of subsets of $\mathbb{N}$ such that $|\mathcal{B}| = \mathfrak{p}$ then there are $\mathfrak{u}_0$ and $\mathfrak{u}_1$ in $\beta\mathbb{N} \setminus \mathbb{N}$ and an amenable subgroup $\mathbb{F} \subseteq \mathbb{S}$ satisfying*

(A) *If $i \in 2$ and $A \in \mathfrak{u}_i$ then for every $B \in \mathcal{B}$ there exists a subgroup $\mathbb{H} \subseteq \mathbb{F}$ of infinite index in $\mathbb{F}$ such that*
$$|\bigcap_{g\in a} B \cap g^{-1}A| = \aleph_0$$
for every finite $a \subseteq \mathbb{F} \setminus \mathbb{H}$.

(B) $\mathfrak{u}_0 \neq \mathfrak{u}_1$

(C) $\mathfrak{u}_0$ *and* $\mathfrak{u}_1$ *are both $\mathbb{F}$-invariant means on $\mathbb{N}$.*

Proof. Let P_0 and P_1 partition $\mathbb{N}$ into infinite sets and let $\{A_\xi\}_{\xi\in\mathfrak{p}}$ be a $\subseteq^*$ increasing family such that $\{P_i \setminus A_\xi\}_{\xi\in\mathfrak{p}}$ generates an ultrafilter $\mathfrak{u}_i$. Let $\{B_\xi\}_{\xi\in\mathfrak{p}}$ enumerate $\mathcal{B}$ so that each set occurs cofinally often. By induction on $\xi \in \mathfrak{p}$ construct $\mathbb{F}_\xi$, $\{\mathbb{G}_{\zeta,\xi}\}_{\zeta\leq\xi}$ and families $\{\mathcal{B}^i_{\eta,\xi}\}_{\eta\in\xi}$ for $i \in 2$ such that:

(1) $\mathbb{F}_\xi$ is a locally finite subgroup of $\mathbb{S}$
(2) the support of each element of $\mathbb{F}_\xi$ is almost contained in A_ξ
(3) $\mathbb{F}_{\eta+1} \setminus \mathbb{G}_{\xi,\eta} \neq \emptyset$ if $\xi \leq \eta \leq \mathfrak{p}$
(4) if $\beta \leq \alpha \leq \gamma \leq \xi$ then
  (a) $\mathbb{F}_\beta \subseteq \mathbb{F}_\alpha \subseteq \mathbb{G}_{\alpha,\gamma}$
  (b) $\mathbb{G}_{\beta,\alpha} \subseteq \mathbb{G}_{\beta,\gamma}$
  (c) $\mathbb{G}_{\beta,\gamma} \cap \mathbb{F}_\xi = \mathbb{G}_{\beta,\xi}$ if $\beta \leq \xi \leq \gamma$
(5) if $i \in 2$ and $\eta \in \xi \in \zeta$ and $B_\eta \cap A_\eta$ is infinite then $B_\eta \cap A_\eta \in \mathcal{B}^i_{\eta,\xi} \subseteq \mathcal{B}^i_{\eta,\zeta}$
(6) each $\mathcal{B}^i_{\eta,\xi}$ is a centred family
(7) if $i \in 2$ and $\xi \in \eta$ and $g \in \mathbb{F}_\eta \setminus \mathbb{G}_{\xi,\eta}$ then $g^{-1}(P_i \cap A_{\eta+1} \setminus A_\xi)$ is in the filter generated by $\mathcal{B}^i_{\xi,\eta}$.

Assuming this induction can be carried out let $\mathbb{F} = \bigcup_{\xi\in\mathfrak{p}} \mathbb{F}_\xi$ and $\mathbb{G}_\xi = \bigcup_{\xi\leq\alpha\in\mathfrak{p}} \mathbb{G}_{\xi,\alpha}$ and note that the index of each $\mathbb{G}_\xi$ in $\mathbb{F}$ is infinite by (3) and (4c). Also $\mathbb{F}$ is amenable by (1).

To see that Condition (A) holds, let $i \in 2$, $A \in \mathfrak{u}_i$ and $B \in \mathcal{B}$. Let $\eta \in \mathfrak{p}$ be so large that $B \cap A_\eta$ is infinite. It may as well be assumed that $A = P_i \setminus A_\eta$ and that $B = B_\eta$. Let $\mathbb{H} = \mathbb{G}_\eta$ and suppose that a is a finite subset $\mathbb{F} \setminus \mathbb{H}$. Let $\beta \in \mathfrak{p}$ be so large $a \subseteq \mathbb{F}_\beta$ and $\eta \leq \beta$. Since (4c) implies that $\mathbb{G}_\eta \cap \mathbb{F}_\beta = \mathbb{G}_{\eta,\beta}$ it follows that a is a finite subset of $\mathbb{F}_\beta \setminus \mathbb{G}_{\eta,\beta}$ and from (5) it follows that $B \in \mathcal{B}^i_{\eta,\beta}$ and from (7) that $g^{-1}(P_i \cap A_{\mu(g)+1} \setminus A_\eta) \in \mathcal{B}^i_{\eta,\mu(g)}$ for each $g \in a$ where $\mu(g)$ is the least ordinal such that $g \in \mathbb{F}_{\mu(g)}$. Since $\eta \leq \mu(g) \leq \beta$ it follows from (5) that $g^{-1}(P_i \cap A_{\mu(g)+1} \setminus A_\eta) \in \mathcal{B}^i_{\eta,\beta}$ for each $g \in a$. Since
$$g^{-1}(P_i \cap A_{\mu(g)+1} \setminus A_\eta) \subseteq^* g^{-1}(P_i \cap A_{\beta+1} \setminus A_\eta)$$

it follows from (6) that

$$B \cap \bigcap_{g \in a} g^{-1}(P_i \cap A_{\beta+1} \setminus A_\eta)$$

is infinite and this is sufficient since $A = P_i \setminus A_\eta \supseteq^* P_i \cap A_{\beta+1} \setminus A_\eta$. To see that Condition (C) holds use (2). The fact that $\mathbb{H} = \mathbb{G}_\eta$ has infinite index has already been established.

To carry out the construction induction will be used to choose for each $\xi \in \mathfrak{p}$ an $(A_\xi, A_{\xi+1})$-involution π_ξ. At the end of the construction $\mathbb{F}_\xi$ will be defined to be the group generated by $\{\pi_\eta\}_{\eta\in\xi} \cup \mathbb{S}_{<\aleph_0}$. This will guarantee that (2) holds. The subgroups $\mathbb{G}_{\eta,\xi} \subseteq \mathbb{F}_\xi$ will be simultaneously constructed induction on ξ. Let $\mathbb{G}_{0,0}$ be the trivial subgroup of $\mathbb{F}_0 = \mathbb{S}_{<\aleph_0}$. If ξ is a limit and $\mathbb{G}_{\zeta,\eta} \subseteq \mathbb{F}_\eta$ have been constructed for $\eta \in \xi$ such that (4c) holds then let $\mathbb{G}_{\zeta,\xi} = \bigcup_{\eta\in\xi} \mathbb{G}_{\zeta,\eta}$. Given $\mathbb{G}_{\zeta,\xi}$ define $\mathbb{G}_{\zeta,\xi+1}$ to be the group generated by

$$\mathbb{G}_{\zeta,\xi} \cup \{g\pi_\xi h \pi_\xi \mid g \in \mathbb{G}_{\zeta,\xi} \text{ and } h \in \mathbb{F}_\xi \setminus \mathbb{S}_{<\aleph_0}\} \tag{3.1}$$

and note that (4c) still holds because the support of $g\pi_\xi h\pi_\xi$ is not contained in A_ξ modulo a finite set provided that $h \notin \mathbb{S}_{<\aleph_0}$.

To construct π_ξ given that $\mathbb{F}_\xi$ and $\{\mathbb{G}_{\zeta,\xi}\}_{\zeta\leq\xi}$ are known let $\mathbb{P}$ be the Polish space of all $(A_\xi, A_{\xi+1})$-involutions under the topology of pointwise convergence. Note that $\mathbb{P}$ has no isolated points.

Claim 1. For any finite $a \subseteq \mathbb{F}_\xi$ and for any infinite $X \subseteq A_\xi$ and for any $i \in 2$ if

$$D_{i,\xi,X} = \left\{ p \in \mathbb{P} \;\middle|\; X \cap \bigcap_{g\in a} gp(P_i \cap A_{\xi+1} \setminus A_\xi) \neq \emptyset \right\}$$

then $D_{i,\xi,X}$ is dense open in $\mathbb{P}$.

Proof. Given a non-empty open set $U \subseteq \mathbb{P}$ there is some $q \in \mathbb{P}$ and a finite $d \subseteq A_{\xi+1}$ such that $\{p \in \mathbb{P} \mid P \supseteq q \restriction d\} \subseteq U$. There are only finitely many $n \in A_\xi$ for which there is some $g \in a$ such that $g(x) \notin A_\xi$. Hence it is possible to find $x \in X \setminus \{g(q(n)) \mid n \in d \text{ and } g \in a\}$ such that $g(x) \in A_\xi$ for each $g \in a$. Let $Z = \{g(x) \mid g \in a\}$ and for each $z \in Z$ let n_z be a distinct element of $A_{\xi+1} \setminus A_\xi$. Then let $\bar{q}$ be a finite involution such that

- $\bar{q} \supseteq q$
- $\bar{q}(n_z) = z$ for each $z \in Z$
- $\mathbf{domain}(\bar{q}) \subseteq A_{\xi+1}$
- $\bar{q}(j) \in A_\xi$ if and only if $j \in A_{\xi+1} \setminus A_\xi$.

Then $\{p \in \mathbb{P} \mid p \supseteq \bar{q}\}$ is open and contained in $D_{i,\xi,X}$.

□

For each $i \in 2$, $\eta \in \xi$ let $D^i_\eta \subseteq A_\eta \subseteq^* A_\xi$ be an infinite set such that $D^i_\eta \subseteq^* B$ for each $B \in \mathcal{B}^i_{\eta,\xi}$. Then use Theorem 3.1 and the fact that $|\xi| < \mathfrak{p}$ applied to $\mathbb{P}$ to find an $(A_\xi, A_{\xi+1})$-involution π_ξ such that for each $\eta \in \xi$, $i \in 2$ and for each finite $a \subseteq \mathbb{F}_\xi$ the set

$$D^i_\eta \cap \bigcap_{g\in a} g^{-1}\pi_\xi(P_i \cap A_{\xi+1} \setminus A_\xi)$$

is infinite. Then for $\eta \in \xi$ let $\mathcal{B}^i_{\eta,\xi+1} = \mathcal{B}^i_{\eta,\xi} \cup \{g^{-1}\pi_\xi(P_i \cap A_{\xi+1} \setminus A_\xi) \mid g \in \mathbb{F}_\xi\}$. By Claim 1 it follows that (6) holds and, since it has already been verified that (2) and (4) hold and (5) holds by construction, it only remains to show that (1), (3) and (7) hold.

Claim 2. If $\xi \in \mathfrak{p}$ and g_1 and g_2 belong to $\mathbb{F}_\xi$ then g_1 and $\pi_\xi g_2 \pi_\xi$ commute modulo $\mathbb{S}_{<\aleph_0}$.

Proof. Note that the support of $\pi_\xi g_2 \pi_\xi$ is almost contained in $A_{\xi+1} \setminus A_\xi$ while the support of g_1 is almost contained in A_ξ. □

Claim 3. If $\xi \in \mathfrak{p}$ and $g \in \mathbb{F}_{\xi+1}$ then g satisfies precisely one of the following assertions:

$$g \in \mathbb{F}_\xi \tag{3.2}$$

$$(\exists g_1 \in \mathbb{F}_\xi)(\exists g_2 \in \mathbb{F}_\xi)\ g = g_1\pi_\xi g_2 \mod \mathbb{S}_{<\aleph_0} \tag{3.3}$$

$$(\exists g_1 \in \mathbb{F}_\xi)(\exists g_2 \in \mathbb{F}_\xi \setminus \mathbb{S}_{<\aleph_0})\ g = g_1\pi_\xi g_2\pi_\xi \mod \mathbb{S}_{<\aleph_0}. \tag{3.4}$$

Proof. Note that if $g = g_1\pi_\xi g_2\pi_\xi \ldots \pi_\xi g_j$ with $j \geq 3$ and $g_i \in \mathbb{F}_\xi \setminus \mathbb{S}_{<\aleph_0}$ for all i such that $1 \leq i \leq j$ then by Claim 2

$$g = g_1 g_3 \pi_\xi g_2 \pi_\xi \pi_\xi g_4 \pi_\xi g_5 \ldots \pi_\xi g_j = g_1^* \pi_\xi g_2^* \pi_\xi g_5 \ldots \pi_\xi g_j$$

where $g_1^* = g_1 g_3$ and $g_2^* = g_2 g_4$. Induction on the length of words now yields that g satisfies at least one of the assertions.

To see that precisely one of the assertions is satisfied it suffices to check that

(8) $g_1\pi_\xi g_2$ moves infinitely many elements of $A_{\xi+1} \setminus A_\xi$

(9) if g_2 in $\mathbb{F}_\xi \setminus \mathbb{S}_{<\aleph_0}$ then $g = g_1\pi_\xi g_2\pi_\xi$ moves infinitely many elements of $A_{\xi+1} \setminus A_\xi$.

Hence it is not possible that $g \equiv g_1\pi_\xi g_2 \mod \mathbb{S}_{<\aleph_0}$ or that $g \equiv g_1\pi_\xi g_2\pi \mod \mathbb{S}_{<\aleph_0}$ if g_2 in $\mathbb{F}_\xi \setminus \mathbb{S}_{<\aleph_0}$. To see that it is not possible that $h_1\pi_\xi h_2 \equiv g_1\pi_\xi g_2\pi \mod \mathbb{S}_{<\aleph_0}$ with g_2 in $\mathbb{F}_\xi \setminus \mathbb{S}_{<\aleph_0}$ observe that Claim 2 establishes that, if this were the case, then there would be g and h such that $g\pi_\xi h \in \mathbb{S}_{<\aleph_0}$ contradicting (8). □

From this it follows that (1) holds. To see this let $a \subseteq \mathbb{F}_{\xi+1}$ be finite. Let

$$a^* = (a \cap \mathbb{F}_\xi) \cup \{g_1, g_2 \in \mathbb{F}_\xi \mid (\exists g \in a)\ g = g_1\pi_\xi g_2\pi_\xi\} \cup \{g_1, g_2 \in \mathbb{F}_\xi \mid (\exists g \in a)\ g = g_1\pi_\xi g_2\}$$

and use the induction hypothesis to let G be the finite subgrouyps generated by a^*. Applying Claim 2 and Claim 3 yields that the group generated by a is contained in the finite set

$$a^* \cup \{g_1\pi_\xi g_2\pi_\xi \mid g_1, g_2 \in a^*\} \cup \{g_1\pi_\xi g_2 \mid g_1, g_2 \in a^*\}.$$

Claim 4. For each $\xi \in \mathfrak{p}$ the index of $\mathbb{G}_\xi$ in $\mathbb{F}$ is uncountable; indeed, if $\xi < \eta < \zeta < \mathfrak{p}$ then $\pi_\eta \not\equiv \pi_\zeta \mod \mathbb{G}_\xi$.

Proof. A routine calculation using Claim 2 shows that if $\xi \leq \theta$ then not only is $\mathbb{G}_{\xi,\theta+1}$ generated by (3.1) but, in fact,

$$\mathbb{G}_{\xi,\theta+1} = \mathbb{G}_{\xi,\theta} \cup \{g\pi_\theta h\pi_\theta \mid g \in \mathbb{G}_{\xi,\theta} \text{ and } h \in \mathbb{F}_\theta \setminus \mathbb{S}_{<\aleph_0}\}. \tag{3.5}$$

It then follows from Claim 3 that $\mathbb{G}_{\xi,\theta+1} \cap \mathbb{F}_\theta = \mathbb{G}_{\xi,\theta}$ and, hence, an inductive argument yields that $\mathbb{G}_\xi \cap \mathbb{F}_\theta = \mathbb{G}_{\xi,\theta}$. Since π_η and π_ζ both belong to $\mathbb{F}_{\zeta+1}$ it follows that, if $\pi_\eta\pi_\zeta^{-1} \in \mathbb{G}_\xi$ then $\pi_\eta\pi_\zeta^{-1} \in \mathbb{G}_\xi \cap \mathbb{F}_{\zeta+1}$ and hence $\pi_\eta\pi_\zeta^{-1} \in \mathbb{G}_{\xi,\zeta+1}$. From 3.5 it follows that either $\pi_\eta\pi_\zeta^{-1} \in \mathbb{G}_{\xi,\zeta}$ or there are $g \in \mathbb{G}_{\xi,\zeta}$ and $h \in \mathbb{F}_\zeta \setminus \mathbb{S}_{<\aleph_0}$ such that $\pi_\eta\pi_\zeta = \pi_\eta\pi_\zeta^{-1} = g\pi_\zeta h\pi_\zeta$. Since $\pi_\zeta \notin \mathbb{F}_\zeta$ the first alternative is not possible. But the second alternative implies that $g\pi_\zeta h = \pi_\eta \in \mathbb{F}_\zeta$ contradicting Claim 3. □

Hence (3) holds and all that needs to be checked is that (7) holds. To establish this proceed by induction on η. The case that $\eta = 1$ is easy and the limit cases are immediate. So assume that the result is true for η and that $\xi < \eta + 1$ and $g \in \mathbb{F}_{\eta+1} \setminus \mathbb{G}_{\xi,\eta+1}$. It must be shown that $g^{-1}(P_i \cap A_{\eta+2} \setminus A_\xi)$ is in the filter generated by $\mathcal{B}^i_{\xi,\eta+1}$. If $g \in \mathbb{F}_\eta \setminus \mathbb{G}_{\xi,\eta+1}$ then it must be that $\xi < \eta$ and the induction hypothesis implies that

$$g^{-1}(P_i \cap A_{\eta+2} \setminus A_\xi) \supseteq^* g^{-1}(P_i \cap A_{\eta+1} \setminus A_\xi) \in \mathcal{B}^i_{\xi,\eta} \subseteq \mathcal{B}^i_{\xi,\eta+1}.$$

Hence assume that $g \notin \mathbb{F}_\eta$.

There are two possibilities by Claim 3.

Case One. $g = g_1\pi_\eta g_2 \mod \mathbb{S}_{<\aleph_0}$ with g_1 and g_2 in $\mathbb{F}_\eta$.

In this case

$(g_1\pi_\eta g_2)^{-1}(P_i\cap A_{\eta+2}\setminus A_\xi) \supseteq^* (g_1\pi_\eta g_2)^{-1}(P_i\cap A_{\eta+2}\setminus A_\eta) \equiv^* g_2^{-1}\pi_\eta(P_i\cap A_{\eta+2}\setminus A_\eta) \supseteq^* g_2^{-1}\pi_\eta(P_i\cap A_{\eta+1}\setminus A_\eta)$
and the result follows from Claim 1 applied to D_{i,ξ,D^i_ξ} and the construction.

Case Two. $g = g_1\pi_\eta g_2\pi_\eta \mod \mathbb{S}_{<\aleph_0}$ with $g_1 \in \mathbb{F}_\eta$ and $g_2 \in \mathbb{F}_\eta \setminus \mathbb{S}_{<\aleph_0}$.

It must be shown that $\pi_\eta g_2\pi_\eta g_1(P_i \cap A_{\eta+2} \setminus A_\xi)$ is in the filter generated by $\mathcal{B}^i_{\xi,\eta+1}$. Notice that since $g \notin \mathbb{G}_{\xi,\eta+1}$ it must be the case that either $g_1 \notin \mathbb{G}_{\xi,\eta}$ or $g_2 \in \mathbb{S}_{<\aleph_0}$. However, the second alternative is ruled out in this case. Using the induction hypothesis it follows that $g_1^{-1}(P_i \cap A_{\theta+1} \setminus A_\xi) \in \mathcal{B}^i_{\theta,\xi}$ for some θ such that $\xi \leq \theta < \eta$. But A_η is almost disjoint from the support of $\pi_\eta g_2^{-1}\pi_\eta$, wheras, A_η contains the range of g_1; so, using Claim 2 it follows that

$$(g_1\pi_\eta g_2\pi_\eta)^{-1}(P_i \cap A_\eta \setminus A_\xi) \supseteq^* g_1^{-1}\pi_\eta g_2^{-1}\pi_\eta(P_i \cap A_\eta \setminus A_\xi) \supseteq^* g_1^{-1}(P_i \cap A_{\theta+1} \setminus A_\xi)$$

and so $(g_1\pi_\eta g_2\pi_\eta)^{-1}(P_i \cap A_{\eta+1} \setminus A_\xi)$ belongs to the ultrafilter generated by $\mathcal{B}^i_{\xi,\eta+1}$.

The last point to observe is that every $\mathbb{F}$-invariant measure on $\mathbb{N}$ is a linear combination of $\mathfrak{u}_1$ and $\mathfrak{u}_2$. To see this let $\mathfrak{m}$ be an $\mathbb{F}$-invariant measure on $\mathbb{N}$. Observe that for any $\xi \in \mathfrak{p}$ the sets

$$\pi_\xi^{-1}(A_\xi), \pi_{\xi+1}^{-1}\pi_\xi^{-1}(A_\xi), \pi_{\xi+2}^{-1}\pi_{\xi+1}^{-1}\pi_\xi^{-1}(A_\xi), \ldots$$

form a pairwise almost disjoint family and hence each must be $\mathfrak{m}$-null. Hence $\mathfrak{m}(P_i \cap A_\xi) = 0$ for each $i \in 2$ and $\xi \in \mathfrak{p}$ and so $\mathfrak{m} = \mathfrak{m}(P_0)\mathfrak{u}_0 + \mathfrak{m}(P_1)\mathfrak{u}_1$. □

Definition 3.3. Given $\mathfrak{u}$ and $\mathfrak{v}$ in $\beta\mathbb{N} \setminus \mathbb{N}$ define a partial order $\prec$ on $\mathfrak{u} \times \mathfrak{v}$ by $(U, V) \prec (U', V')$ if $U \subseteq^* U'$ and $V \subseteq^* V'$. A family

$$\{x_a\}_{a\in\mathfrak{u}\times\mathfrak{v}} \subseteq \ell^*_\infty(G)$$

will be said to converge if it converges as a net with respect to the directed partial order $\prec$ in the weak*-topology.

Corollary 3.1. *Under the hypothesis of Lemma 3.1, if $\mathfrak{n} \in \beta\mathbb{N} \setminus \mathbb{N}$ is generated by a set of cardinality $\mathfrak{p}$ then there is an amenable group $\mathbb{F} \subseteq \mathbb{S}$ and distinct $\mathfrak{u}_1$ and $\mathfrak{u}_2$ in $\beta\mathbb{N} \setminus \mathbb{N}$ such that for any $\mathfrak{m}$ that is an $\mathbb{F}$-invariant mean there are $n_{i,A,B} \in \beta\mathbb{N} \setminus \mathbb{N}$ such that*

- $\{\mathfrak{n}_{i,A,B}\}_{(A,B)\in\mathfrak{u}_i\times\mathfrak{n}}$ *converges to* $\mathfrak{n}$
- $\{\mathfrak{m}\Diamond\mathfrak{n}_{i,A,B}\}_{(A,B)\in\mathfrak{u}_i\times\mathfrak{n}}$ *converges to* $\mathfrak{u}_i$.

In particular, the mapping $x \mapsto \mathfrak{m}\Diamond x$ is not weak continuous at $\mathfrak{n}$.*

Proof. Let $\mathcal{B} \subseteq \mathfrak{n}$ be a generating set of cardinality $\mathfrak{p}$ and let $\mathfrak{F}$ and $\mathfrak{u}_1$ and $\mathfrak{u}_2$ satisfy the conclusion of Lemma 3.1 for $\mathcal{B}$. Using (A), for each $i \in 2$, $A \in \mathfrak{u}_i$ and $B \in \mathcal{B}$ let $\mathbb{H}_{i,A,B}$ be a subgroup of $\mathbb{F}$ of infinite index such that

$$\{B \cap g^{-1}A \mid g \in \mathbb{F} \setminus \mathbb{H}_{i,A,B}\}$$

can be extended to an ultrafilter $\mathfrak{n}_{i,A,B}$. The choice of $\mathcal{B}$ guarantees that $\{\mathfrak{n}_{i,A,B}\}_{(A,B)\in\mathfrak{u}_i\times\mathfrak{n}}$ converges to $\mathfrak{n}$ for each $i \in 2$.

Since each $\mathbb{H}_{i,A,B}$ has infinite index it follows that $\mathfrak{m}(\mathbb{H}_{i,A,B}) = 0$ for all i, A and B. Moreover, for $i \in 2$ and $A \in \mathfrak{u}_i$

$$(\forall g \in \mathbb{F} \setminus \mathbb{H}_{i,A,B})\ g^{-1}A \in \mathfrak{n}_{i,A,B}$$

for all $B \in \mathfrak{n}$ and, hence,

$$\int\int 1_A(gk)d\mathfrak{n}_{i,A,B}(k)d\mathfrak{m}(g) = 1$$

or any $B \in \mathfrak{n}$. Therefore $\{\mathfrak{n}_{i,A,B}\}_{(A,B)\in\mathfrak{u}_i\times\mathfrak{n}}$ converges to $\mathfrak{u}_i$ for each $i \in 2$. □

Corollary 3.2. *Assuming the Continuum Hypothesis, there is an amenable group $\mathbb{F} \subseteq \mathbb{S}$ and distinct $\mathfrak{u}_1$ and $\mathfrak{u}_2$ in $\beta\mathbb{N} \setminus \mathbb{N}$ such that for any $\mathfrak{m}$ that is an $\mathbb{F}$-invariant mean and for any $\mathfrak{n} \in \beta\mathbb{N} \setminus \mathbb{N}$ there are $n_{i,A,B} \in \beta\mathbb{N} \setminus \mathbb{N}$ such that*

- $\{\mathfrak{n}_{i,A,B}\}_{(A,B)\in\mathfrak{u}_i\times\mathfrak{n}}$ *converges to* $\mathfrak{n}$
- $\{\mathfrak{m}\Diamond\mathfrak{n}_{i,A,B}\}_{(A,B)\in\mathfrak{u}_i\times\mathfrak{n}}$ *converges to* $\mathfrak{u}_i$.

In particular, the mapping $x \mapsto \mathfrak{m}\Diamond x$ *is not weak* continuous at any point of* $\beta\mathbb{N}\setminus\mathbb{N}$.

Proof. Let $\mathcal{B}$ be the set of all infinite subsets of $\mathbb{N}$. Under the Continuum Hypothesis $\mathfrak{p} = \aleph_1$ and there are tower ultrafilters of cardinality $\aleph_1$ so the argument of Corollary 3.1 can be applied. To obtain the conclusion note that if $\mathfrak{n} \in \beta\mathbb{N}\setminus\mathbb{N}$ then $\mathfrak{n} \subseteq \mathcal{B}$. □

4. Remarks and questions

It should be noted that there are many examples of models of set theory in which the hypothesis of Lemma 3.1 holds yet the Continuum Hypothesis fails. The model obtained by iteratively adding ω_2 Sacks reals with countable support to a model of the Continuum Hypothesis provides a prototypical example. On the other hand, the model obtained by adding $\aleph_2$ Cohen reals to a model of the Continuum Hypothesis yields a model where $\mathfrak{p} = \aleph_1$ and there are no tower ultrafilters. But even more is true in this model.

Proposition 4.1. *M. Foreman In the model obtained by adding* $\aleph_2$ *Cohen reals to a model of the Continuum Hypothesis there is no locally finite subgroup of* $\mathbb{S}$ *such that the set of means invariant under the natural action of this group on* $\mathbb{N}$ *is finite dimensional.*

Proof. This follows from a simple modification of the argument of [2], so familiarity with the proof of Theorem 4.1 from [2] will be assumed. Instead of defining $c_\alpha = \{n \mid p(\alpha, n) = 1\}$ as in the second sentence of the proof of Theorem 4.1, define Cohen forcing $\mathbb{P}$ to be all partial functions from $\kappa\times\mathbb{N}$ to $\mathbb{N}$ with finite domain and let $c_{\alpha,j} = \{n \mid p(\alpha, n) = j\}$.

Now suppose that $\dot{G}$ is a $\mathbb{P}$ name for a locally finite group and that

$$1 \Vdash_{\mathbb{P}} \text{“the set of means invariant under } \dot{G} \text{ is contained in the linear span of } \{\mathfrak{m}_i\}_{i=1}^k\text{”}.$$

For each $\alpha \in \kappa$ it is possible to find $j(\alpha) \in \mathbb{N}$ and $p_\alpha \in \mathbb{P}$ such that $p_\alpha \Vdash_{\mathbb{P}}$ “$(\forall i \le k)\ \mathfrak{m}_j(c_{\alpha,j(\alpha)}) < 1$”. Using Proposition 1 of [7] (see also the Lemma of Yang on page 11 of [2]) it follows that for each α there is $r_\alpha < 1$ and $p^*_\alpha \supseteq p_\alpha$ such that $p^*_\alpha \Vdash_{\mathbb{P}}$ “$c_{\alpha,j(\alpha)}$ is not r_α thick”. The argument of [2] can nw be applied to yield a contradiction. □

It is important to observe that the argument of Proposition 4.1 applies only to locally finite groups.

Question 4.1. Is it the case that every non-trivial element $\mathfrak{m}$ of $\Lambda_1(\odot)$ for the action $\odot$ of $\mathbb{F}$ ithe function $x \mapsto \mathfrak{m} \odot_1 x$ is the finite linear sum of constant functions and identity functions?

Question 4.2. In the model obtained by adding $\aleph_2$ Cohen reals to a model of the Continuum Hypothesis no locally finite subgroup of the full symmetric group acts with a unique invariant mean. Is it true that in this model $\Lambda_1(\odot)$ is trivial for the action of any locally finite subgroup of the full symmetric group? What about every amenable group?

Question 4.3. Is there a model where there is an amenable subgroup of the full symmetric group acting on $\mathbb{N}$ with a $k+1$ dimensional subspace of invariant means, but there is no such action of an amenable group with a k dimensional subspace of invariant means?

Question 4.4. Given an amenable group $\mathbb{G}$ with a subgroup $\mathbb{H}$ let $S_{\mathbb{H}}$ denote the left cosets of $\mathbb{H}$ and let $\odot$ denote the natural left action of $\mathbb{G}$ on $S_{\mathbb{H}}$. Is it possible for $\odot_1$ to have a unique invariant mean?

Question 4.5. Is there a characterization of the actions $\odot$ of groups on sets for which $\Lambda_1(\odot)$ is trivial?

Question 4.6. What can be said about $\Lambda_2(\odot)$?

As explained in the introduction, this article has focussed on actions of amenable groups because of their relevance to Proposition 2.2. However, it may be of interest to look at the topological centres of actions of non-amenable groups as well. While it is easy to construct for $\mathfrak{u} \in \beta\mathbb{N} \setminus \mathbb{N}$ a subgroup $\mathbb{G} \subseteq \mathbb{S}$ such that $\mathfrak{u}$ is the unique invariant mean for the action of $\mathbb{G}$ — simply let $\mathbb{G}$ be all permutations that preserve $\mathfrak{u}$ — the following result from [6] is less obvious.

Theorem 4.1 (E. van Douwen)**.** *There is a transitive action $\odot$ of $\mathbb{F}_2$, the free group of rank two, on a countable set such that each non-identity element of $\mathbb{F}_2$ fixes only finitely many points, yet $\odot$ admits a finitely additive $\mathbb{F}_2$ invariant element of ℓ_∞^*.*

There are a number of other examples in [6] for which it might be of interest to calculate the topological centres.

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