COMPLEXITY OF WEAKLY ALMOST PERIODIC FUNCTIONS

JURIS STEPRĀNS

1. Introduction

Given a topological group G let $C(G)$ denote the Banach space of bounded, continous real valued function on G. Eberlein [1] defined a function $f \in C(G)$ to be weakly almost periodic if the weak closure of all of its translates is compact in the weak topology on $C(G)$ — in other words, if $f_x(y)$ is defined to be $f(yx^{-1})$ then the weak closure of $\{f_x \mid x \in G\}$ is weakly compact. The set of weakly almost periodic function on G will be denoted by WAP(G).

In the case that G is a countable discrete group $C(G) = \ell_\infty(G)$. If G is infinite then $\ell_\infty(G)$ is not separable, but considered with the pointwise topology $\ell_\infty(G)$ becomes a Polish space and it is natural to ask for the descriptive set theoretic complexity of WAP(G) as a subset of $\ell_\infty(G)$. It has been shown [4, 5] that if G is the countable Boolean group then WAP(G) is a non-Borel Π^1_1 set. However, the question of whether or not WAP($\mathbb{Z}$) is a non-Borel Π^1_1 set was not solved by the methods of [4, 5]. The goal of the current work is to show that not only is WAP($\mathbb{Z}$) a non-Borel Π^1_1, but it is also Π^1_1-complete.

Recall that a subset A of a zero-dimensional Polish space X is Π^1_1-complete if it is Π^1_1 and if for every other zero-dimensional Polish space Y and $B \subseteq Y$ that is Π^1_1 there is a continuous function $f : Y \to X$ such that $B = f^{-1}A$. In other words, no Π^1_1 set is more complicated than A.

Theorem 1.1 (Grothendieck [2]). *Given a semigroup $(G, \cdot)$, a function $f \in C(G)$ is in WAP(G) if and only if whenever $\{x_i\}_{i=0}^\infty$ and $\{y_i\}_{i=0}^\infty$ are two sequences in G*

$$\lim_{i\to\infty}\lim_{k\to\infty} f(x_i \cdot y_k) = \lim_{k\to\infty}\lim_{i\to\infty} f(x_i \cdot y_k)$$

provided that both limits exist.

Definition 1.1. A cancellative abelian semigroup $(G, +)$ will be said to be $\mathbb{N}$-*like* if there is an irreflexive, transitive relation $\prec$ on G such that:

(1) for any one-to-one sequence $\{x_i\}_{i\in\omega}$ from G there is a sequence $\{z_i\}_{i\in\omega}$ from G such that
  - $z_i \prec z_{i+1}$ for each $i \in \omega$
  - for each $k \in \omega$ there is some $j \in \omega$ such that $\sum_{i=0}^k z_i = x_j$

(2) for any $g \in G$ there are only finitely many sequences $\{a_j\}_{j=1}^k$ such that $a_1 \prec a_2 \prec \ldots \prec a_k$ and such that $\sum_{j=1}^k a_j = g$

(3) for any $g \in G$ there are infinitely many $a \in G$ such that $g \prec a$.

The triple $(G, +, \prec)$ will be called an $\mathbb{N}$-like semigroup if $(G, +)$ is an abelian semigroup and the partial order $\prec$ on G witnesses that $(G, +)$ is $\mathbb{N}$-like.

Observe that the natural relation $\lneq$ of $\mathbb{N}$ witnesses that $(\mathbb{N}, +)$ is $\mathbb{N}$-like. However, this natural relation of $\mathbb{Z}$ does not witness that $(\mathbb{Z}, +)$ is $\mathbb{N}$-like. Nevertheless, the relation $\prec$ defined by $a \prec b$ if $0 \leq a \lneq b$ or $b \lneq a \leq 0$ does witness this.

Definition 1.2. Let $(G, +, \prec)$ be an $\mathbb{N}$-like semigroup. Given $X \subseteq G$ and $Y \subseteq G$ define the tree $\mathbb{T}_{X,Y}$ on $G \times G$ to consist of all sequences $s \in (G \times G)^{<\omega}$ such that, letting $s(n) = (s_0(n), s_1(n))$:

- $s_0(n) \prec s_0(n+1)$ and $s_1(n) \prec s_1(n+1)$

- if $\sigma_i^s(n)$ is defined to $\sum_{k=0}^n s_j(k)$ then the following hold for each n:

$$\sigma_1^s(n) \in \bigcap_{j\in n} Y - \sigma_0^s(j)$$

$$\sigma_0^s(n) \in \bigcap_{j\in n} X - \sigma_1^s(j).$$

where $\sigma \in \bigcap_{j\in n} Y - \sigma_j$ will be used as an abbreviation for $(\forall j \in n)\ \sigma + \sigma_j \in Y$. Define $\mathbb{T}_X = \mathbb{T}_{X,\mathbb{Z}\setminus X}$.

Lemma 1.3. *Let $(G, +, \prec)$ be an $\mathbb{N}$-like semigroup and $X \subseteq G$. The tree $\mathbb{T}_X$ is ill-founded if and only if $\chi_X \in WAP(G)$.*

Proof. If $\mathbb{T}_X$ is ill-founded let b be an infinite branch of $\mathbb{T}_X$ and let $\sigma_j^b(i) = \sum_{\ell\leq i} b_j(\ell)$. Then

$$\lim_i \lim_k \chi_X(\sigma_1^b(i) + \sigma_0^b(k)) = 1$$

while

$$\lim_k \lim_i \chi(\sigma_1^b(i) + \sigma_0^b(k)) = 0.$$

On the other hand, suppose there are sequences $\{x_i\}_{i=0}^\infty$ and $\{y_i\}_{i=0}^\infty$ in G such that

$$\lim_i \lim_k \chi_X(x_i + y_k) \neq \lim_k \lim_i \chi_X(x_i + y_k).$$

Without loss of generality, $\lim_i \lim_k \chi_X(x_i + y_k) = 1$ and $\lim_k \lim_i \chi_X(x_i + y_k) = 0$. Begin by choosing subsequences $\{\bar{x}_i\}_{i=0}^\infty \subseteq \{x_i\}_{i=0}^\infty$ and $\{\bar{y}_i\}_{i=0}^\infty \subseteq \{y_i\}_{i=0}^\infty$ such that

$$(\forall i)(\forall k > i)\ \bar{x}_i + \bar{y}_k \in X$$

$$(\forall k)(\forall i > k)\ \bar{x}_i + \bar{y}_k \notin X$$

and then use Condition (1) of Definition 1.1 to find $\{\bar{b}(i)\}_{i\in\omega}$ such that for each k there is $j_0(k)$ such that $\sum_{i=0}^k \bar{b}(i) = \bar{x}_{j_0(k)}$. Then apply Condition (1) of Definition 1.1 to find $\{b_1(i)\}_{i\in\omega}$ such that for each k there is $s(k)$ such that $j_1(k) = j_0(s(k))$ and $\sum_{i=0}^k b_1(i) = \bar{y}_{j_1(k)}$ Letting $b_0(0) = \sum_{\ell=0}^{s(0)} \bar{b}(\ell)$ and $b_0(i+1) = \sum_{\ell=s(i)+1}^{s(i+1)} \bar{b}(\ell)$ so that defining $b(i) = (b_0(i), b_1(i))$ yields an infinite branch through $\mathbb{T}_X$. □

Notation 1.4. For t and s in $(G \times G)^{<\omega}$ define $s \sqsubseteq t$ if

$$\{s_1(j) \mid j \in |s|\} \subseteq \left\{ \sum_{j=k_1}^{k_2} t_1(j) \mid k_1 \leq k_2 < |t| \right\}.$$

Lemma 1.5. *Let $(G, +, \prec)$ be an $\mathbb{N}$-like semigroup and $X \subseteq G$. If $T \subseteq \mathbb{T}_X$ is a well-founded subtree of $\mathbb{T}_X$ then there is no one-to-one function $b : \omega \to G \times G$ such that for all k there is $t \in T$ such that $b \restriction k \sqsubseteq t$*

Proof. Note that by Condition (2) of Definition 1.1, for any $g \in G$ the set of all $t \in \mathbb{T}_X$ such that $g = \sum_{j=m}^{|t|-1} t_1(j)$ is finite. Let L_k be the finite set of $t \in T$ such that $b_1(k) = \sum_{j=m}^{|t|-1} t_1(j)$ for some $m \leq |t| - 1$. If $b \restriction k \sqsubseteq t$ for some $t \in T$ then for all $m \in k$ there is some j such that $t \restriction j \in L_m$. Now apply compactness to get a contradiction to the well-foundedness of T. □

Notation 1.6. Define $\sqcap(t, Z) = \bigcap_{i\in|t|} Z - \sigma_0^t(i) = \{g \in G \mid (\forall i \in |t|)\ g + \sigma_0^t(i) \in Z\}$.

Lemma 1.7. *Let $(G, +, \prec)$ be an $\mathbb{N}$-like semigroup. Let Tr denote the set of all subtrees of $\mathbb{N}^{<\omega}$ and $\mathcal{P}(G)$ denote the subsets of G with both considered with the pointwise topology. There is a continuous mapping $\Psi : Tr \to \mathcal{P}(G)$ such that T is well founded if and only if $\chi_{\Psi(X)}$ is in $WAP(G)$.*

Proof. Let $\{g_i\}_{i\in\omega}$ enumerate G and let $\{t_i\}_{i\in\omega}$ enumerate $(G\times G)^{<\omega}$. Let $\{\tau_n\}_{n\in\omega}$ enumerate $\mathbb{N}^{<\omega}$ according to the lexicographic ordering so that, in particular, $T_n=\{\tau_i \mid i\in n\}$ is closed under initial segments for each n and, hence, is itself a tree.

Given any tree $T\subseteq\mathbb{N}^{<\omega}$ define $\Psi(T)$ by constructing by induction isomorphisms $\psi_n : T_n\cap T\to (G\times G)^{<\omega}$ and A_n and B_n such that:

(1) $A_n\in[G]^{<\aleph_0}$ and $B_n\in[G]^{<\aleph_0}$
(2) $\{g_i\}_{i\in n}\subseteq A_n\cup B_n$
(3) $A_n\cap B_n=\emptyset$
(4) $A_n\subseteq A_{n+1}$ and $B_n\subseteq B_{n+1}$
(5) $\psi_n\subseteq\psi_{n+1}$
(6) if $i\le n$ is such that $t_i\not\sqsubseteq t$ for any $t\in T_n$ then $\sqcap(t_i,B_n)=\sqcap(t_i,B_i)$
(7) if $i\le n$ is such that τ_i is maximal in T then $\sqcap(\psi_n(\tau_i),B_n)=\sqcap(\psi_n(\tau_i),B_i)$
(8) $\psi_n(\tau_j)\in\mathbb{T}_{A_n,B_n}$ for each $j\le n$
(9) $\psi_n(\tau)\not\sqsubseteq\psi_n(\tau')$ if τ and τ' are in T_n and are incomparable.

If this induction can be completed then let $\Psi(T)=\bigcup_n A_n$ and let $\psi=\bigcup_n\psi_n$. It is clear that $\Psi(T)\cap\{g_i\}_{i\in n}$ depends only on A_n and B_n and, as will be seen, these two sets depend only on $T_n\cap T$. Hence Ψ is continuous.

To complete the induction suppose that $\psi_n : T_n\cap T\to(\mathbb{N}\times\mathbb{N})^{<\omega}$ and A_n and B_n are given. If $\tau_{n+1}\notin T$ then let $B_{n+1}=B_n$ and $\psi_{n+1}=\psi_n$. If $g_n\notin B_n$ then let $A_{n+1}=A_n\cup\{g_n\}$; otherwise let $A_{n+1}=A_n$. On the other hand, if $\tau_{n+1}\in T$ then let $j\le n$ be such that $\tau_{n+1}=\tau_j{}^\frown u$ for some $u\in\mathbb{N}$. Begin by using Induction Hypothesis (3) and the fact that $(G,+)$ is cancellative to let a be the least element in the enumeration of G such that $\psi_n(\tau_j)_1(|\tau_j|-1)\prec a$ and such that $a+\sigma_0^{\psi_n(\tau_j)}(\ell)\notin B_n$ for all $\ell\in|\tau_j|$ and let

$$A_{n+1}=\begin{cases}A_n\cup\left\{a+\sigma_0^{\psi_n(\tau_j)}(\ell)\ \mid \ell\in|\tau_j|\right\} & \text{if } g_n\in B_n\\ A_n\cup\left\{a+\sigma_0^{\psi_n(\tau_j)}(\ell)\ \mid \ell\in|\tau_j|\right\}\cup\{g_n\} & \text{otherwise.}\end{cases}$$

Once again using Induction Hypothesis (3) and the fact that $(G,+)$ is cancellative let b be the least element in the enumeration of G such that $\psi_n(\tau_j)_0(|\tau_j|-1)\prec b$ and such that

$$(\forall\ell\in|\tau_j|)\ b+\sigma_1^{\psi_n(\tau_j)}(\ell)\notin A_{n+1} \tag{1.1}$$

$$(\forall x\in B_n)(\forall i\le n)(\forall k<|t_i|)(\forall\ell<|\tau_j|)\ b\ne x+\sigma_1^{t_i}(k+1)+\sigma_1^{t_i}(k)+\sigma_1^{\psi_n(\tau_j)}(\ell) \tag{1.2}$$

$$(\forall x\in B_n)(\forall i\le n)(\forall k<|t_i|)(\forall\ell<|\tau_j|)\ b\ne x+\sigma_1^{\psi_n(\tau_i)}(k+1)+\sigma_1^{\psi_n(\tau_i)}(k)+\sigma_1^{\psi_n(\tau_j)}(\ell) \tag{1.3}$$

and such that Induction Hypotheses (9) continues to hold. Then let $B_{n+1}=B_n\cup\left\{b+\sigma_1^{\psi_n(\tau_j)}(\ell)\ \mid \ell\in|\tau_j|\right\}$. Let

$$\psi_{n+1}(\tau_{n+1})=\psi_n(\tau_j)^\frown\left(\sigma_0^{\psi_n(\tau_j)}(|\tau_j|-1)+a,\sigma_1^{\psi_n(\tau_j)}(|\tau_j|-1)+b\right).$$

It is clear that Induction Hypotheses (1) to (5) and (8) and (9) are all satisfied. It is also clear A_{n+1} and B_{n+1} depend only on $T_{n+1}\cap T$.

To see that Induction Hypothesis (6) holds suppose that $i\le n$ and that $w\in\sqcap(t_i,B_{n+1})\setminus\sqcap(t_i,B_i)$. It follows from the induction hypothesis that

$$\sqcap(t_i,B_n)=\sqcap(t_i,B_i)$$

so it may be supposed that $w \in \sqcap(t_i, B_{n+1}) \setminus \sqcap(t_i, B_n)$. If that $t_i \sqsubseteq \psi_{n+1}(\tau_{n+1})$ then Induction Hypothesis (6) is not relevant so assume that $t_i \not\sqsubseteq \psi_{n+1}(\tau_{n+1})$ and let $k \in |t_i|$ be such that

$$(\forall k_1 \leq k_2 < |\tau_{n+1}|)\ t_i(k) \neq \sum_{\ell=k_1}^{k_2} \psi_{n+1}(\tau_{n+1})_1(\ell) \tag{1.4}$$

It follows that

$$w = x + \sigma_1^{t_i}(k) = x' + \sigma_1^{t_i}(k+1)$$

for x and x' in B_{n+1}. It cannot be the case that both x and x' belong to B_n so consider first the case that $x \in B_n$ and $x' \in B_{n+1} \setminus B_n$. Then $x' = b + \sigma_1^{\psi_n(\tau_j)}(\ell)$ for some $\ell \in |\tau_j|$ and solving for b contradicts Inequality (1.2). A similar contradiction holds if $x' \in B_n$ and $x \in B_{n+1} \setminus B_n$ Hence

$$b + \sigma_1^{\psi_n(\tau_j)}(\ell) + \sigma_1^{t_i}(k) = b + \sigma_1^{\psi_n(\tau_j)}(\ell') + \sigma_1^{t_i}(k+1)$$

and hence

$$\sum_{u=\ell'}^{\ell} \psi_{n+1}(\tau_{n+1})_1(u) = \sum_{u=\ell'}^{\ell} \psi_n(\tau_j)_1(u) = \sigma_1^{t_i}(k+1) - \sigma_1^{t_i}(k) = t_i(k)$$

contradicting Inequality 1.4.

To see that Induction Hypothesis (7) holds, suppose that $i < n+1$ is such that τ_i is maximal in T. Observe that since $\psi_n(\tau_i)$ is maximal in $\psi_n(T_n)$ it follows that there is a maximal $k < |\tau_i| - 1$ such that $\psi_n(\tau_i) \restriction k = \psi_n(\tau_j) \restriction k$ and by Induction Hypothesis (9)

$$(\forall k_1 \leq k_2 < |\tau_{n+1}|)\ \psi_n(\tau_i)(k) \neq \sum_{\ell=k_1}^{k_2} \psi_{n+1}(\tau_{n+1})_1(\ell). \tag{1.5}$$

As before, it may be supposed that $w \in \sqcap(\psi_n(\tau_i), B_{n+1}) \setminus \sqcap(\psi_n(\tau_i), B_n)$. It follows that

$$w = x + \sigma_1^{\psi_n(\tau_i)}(k) = x' + \sigma_1^{\psi_n(\tau_i)}(k+1)$$

for x and x' in B_{n+1}. It cannot be the case that both x and x' belong to B_n. The possibility that $x \in B_n$ and $x' \in B_{n+1} \setminus B_n$ or that $x' \in B_n$ and $x \in B_{n+1} \setminus B_n$ is ruled out by Inequality (1.3). Hence

$$b + \sigma_1^{\psi_n(\tau_j)}(\ell) + \sigma_1^{\psi_n(\tau_i)}(k) = b + \sigma_1^{\psi_n(\tau_j)}(\ell') + \sigma_1^{\psi_n(\tau_i)}(k+1)$$

and hence

$$\sum_{u=\ell'}^{\ell} \psi_{n+1}(\tau_{n+1})_1(u) = \sum_{u=\ell'}^{\ell} \psi_n(\tau_j)_1(u) = \sigma_1^{\psi_n(\tau_i)}(k+1) - \sigma_1^{\psi_n(\tau_k)}(k) = \psi_{n+1}(\tau_i)(k)$$

contradicting Inequality (1.5).

It needs to be shown that $\mathbb{T}_{\Psi(T)}$ is well founded if and only if T is. To see this, suppose first that T is well founded and that $b : \omega \to G \times G$ is a branch through $\mathbb{T}_{\Psi(T)}$. There are two cases to consider, the first being that there is some $k \in \omega$ such that $b \restriction k \not\sqsubseteq t$ for any $t \in \psi(T)$. In this case there is some $i \in \omega$ such that $b \restriction k = t_i$ and it follows that Induction Hypothesis (6) holds for all $n \geq i$. Therefore $\sqcap(t_i, B_n) = \sqcap(t_i, B_i)$ for all $n \geq i$. Since $\sqcap(t_i, B_i)$ is finite there is some $j > k$ such that $b_1(j) \notin \sqcap(t_i, B_i)$. However, if $b \restriction j+1 \in \mathbb{T}_{\Psi(T)}$ then it must be that

$$b_1(j) \in \bigcap_{\ell \in j} (G \setminus \Psi(T)) - \sigma_0^{b\restriction j+1}(\ell) \subseteq \bigcap_{\ell \in k} (G \setminus \Psi(T)) - \sigma_0^{b\restriction j+1}(\ell)$$

and hence that there is some J such that

$$b_1(j) \in \bigcap_{\ell \in k} B_J - \sigma_0^{b\restriction j+1}(\ell) = \sqcap(b \restriction k, B_J) = \sqcap(t_i, B_J) = \sqcap(t_i, B_i)$$

yielding a contradiction. Hence it must be that for all $k \in \omega$ there is some $t \in \psi(T)$ such that $b \restriction k \sqsubseteq t$. Then apply Lemma 1.5 to conclude that $\psi(T)$ is not well founded contradicting that ψ is an isomorphism.

To show that $\mathbb{T}_{\Psi(T)}$ is not well founded if T is not well founded it suffices to show that $\psi(T) \subseteq \mathbb{T}_{\Psi(T)}$. This follows from Induction Hypothesis (8) since if $\tau \in T$ then there is some $n \in \omega$ such that $\tau = \tau_n$. Then $\psi_n(\tau_j) \in \mathbb{T}_{A_n,B_n} \subseteq \mathbb{T}_{\Psi(T)}$.

□

Corollary 1.8. *Both WAP*$(\mathbb{N})$ *and WAP*$(\mathbb{Z})$ *are* Π^1_1*-complete.*

Proof. This is a restatement of Lemma 1.7 for thein the language of Wadge reducibility. See Definition 21.13 and Theorem 27.1 of [3]. □

The following corollary yields as a special the case the result of [4, 5] for the Boolean group.

Corollary 1.9. *If G is the direct sum of finite abelian groups then WAP*(G) *is* Π^1_1*-complete.*

Proof. Let $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}_{k_i}$. For $g \in G$ let $m(g)$ be the maximum element of the support of g for all g other than the indentity. Define $g \prec h$ if and only if $m(g) \lneq m(h)$. It is routine to see that $(G, +, \prec)$ is $\mathbb{N}$-like. Then apply Lemma 1.7. □

Corollary 1.10. *If $\mathbb{Q}$ is considered as an additive group then WAP*$(\mathbb{Q})$ *is* Π^1_1*-complete.*

Proof. Lemma 1.7 will be applied of course. Given rationals p and q such that $p = m_p/n_p$ and $q = m_q/n_q$ with m_p, n_p, m_q and n_q all integers an n_p and n_q minimal positive define $q \prec p$ if one of the following conditions holds:

- $n_p = n_q$ and $q \geq 0$ and $p - q > 1$
- $n_p = n_q$ and $q \leq 0$ and $p - q < -1$
- $n_p \gneq n_q$ and $\text{sign}(p) = \text{sign}(q)$.

To see that $(\mathbb{Q}, +, \prec)$ is $\mathbb{N}$-like note first that $\prec$ is transitive and irreflexive. Condition (3) obviously holds. Suppose that $\{x_i\}_{i\in\omega}$ is a one-to-one sequence of element of $\mathbb{Q}$. Choose an infinite $X \subseteq \omega$ such that $\text{sign}(x_i)$ is the same for all $i \in X$ and

$$(\exists r \in \mathbb{R} \cup \{-\infty, \infty\}) \lim_{i\in X} x_i = r \tag{1.6}$$

Let $x_i = m_i/n_i$ with n_i and m_i co-prime and $i < j$ then $n_j > n_i$. If there is some n such that $n_i = n$ for infinitely many $i \in X$ then it is easy to verify that Condition (1) holds. Otherwise there is an infinite subset $Z \subseteq X$ such that n_i is strictly increasing on Z. Once again it is easy to verify that Condition (1) holds.

To see that Condition (2) holds note that if $0 \leq q = m/n \in \mathbb{Q}$ there are only finitely many $p = \bar{m}/\bar{n}$ such that $0 \leq p \leq q$ and $\bar{n} \leq n$. A similar claim holds if $q \leq 0$. From this Condition (2) follows easily. □

As a final remark it is worth noting that some form of cancellation is needed in the semigroup G. To see this consider the semigroup $(\mathbb{Q}, \max)$.

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Department of Mathematics, York University, 4700 Keele Street, Toronto, Ontario, Canada M3J 1P3

E-mail address: steprans@yorku.ca