Universal graphs and functions on ω_1

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Abstract

It is shown to be consistent with various values of $\mathfrak{b}$ and $\mathfrak{d}$ that there is a universal graph on ω_1. Moreover, it is also shown that it is consistent that there is a universal graph on ω_1 — in other words, a universal symmetric function from ω_1^2 to 2 — but no such function from ω_1^2 to ω. The method used relies on iterating well known reals, such as Miller and Laver reals, and alternating this with the PID forcing which adds no new reals.

1. Introduction

1.1. Background to the problem

It is well known that countably saturated models are universal for models of cardinality $\aleph_1$. However, in the case of graphs, the existence of a saturated model is equivalent to $2^{\aleph_0} = \aleph_1$. These two observations immediately raise the question of whether it is possible to have a universal graph of cardinality $\aleph_1$ in the absence of the Continuum Hypothesis. This provided the motivation for the articles [1], [2] and [3] which solved not only this question, but also others about the universality of different structures.

While the results to be presented here have their roots in this work, they are also motivated by considerations that are not model theoretic. A function $U : X \times X \to X$ is said to be *(Sierpiński) universal* if for any $G : X \times X \to X$ there is $e : X \to X$ such that $G(x, y) = U(e(x), e(y))$ for all x and y in X. The function e will be called an *embedding* of G into U. An early reference to this notion can be found in Problem 132 of the Scottish Book [4] in which Sierpiński asked if there is a Borel function which is universal in this sense, when X is the real line. He had already shown in [5] that there is a Borel universal function assuming the Continuum Hypothesis. This notion of universal function is also studied in Rado [6]. More recently, this notion and various generalizations of it were studied in [7], in which a restricted form of the following definition appears as Definition 7.5.

Definition 1.1. A function $U : \kappa \times \kappa \to \lambda$ is *weakly universal* if for every $f : \kappa \times \kappa \to \lambda$ there exist one-to-one functions $h : \kappa \to \kappa$ and $k : \lambda \to \lambda$ such that $k(f(\alpha, \beta)) = U(h(\alpha), h(\beta))$ for all α and β in κ. The pair (h, k) will be called a weak embedding.

Definition 7.4 of [7] defines a function $U : \kappa \times \kappa \to \kappa$ to be *model theoretically universal* if it is weakly universal, as in Definition 1.1, but with $h = k$. Note that U is Sierpiński universal if it is weakly universal with k being the identity. Remark 7.7 of [7] claims that all three notions — Sierpiński universal, weakly universal, and model theoretically universal — are equivalent for maps into 2.

The following is Theorem 5.9 of [7] showing that there is no difference between asking about the existence of universal graphs — in other words, symmetric, irreflexive functions from ω_1^2 to 2 — and non-symmetric functions from ω_1^2 to 2.

Theorem 1.2. *For any infinite cardinal κ the following are equivalent:*

1. *For each $n \in \mathbb{N}$ there is a universal function from $\kappa \times \kappa$ to n.*

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2. *For some $n \in \mathbb{N}$ with $n \geq 2$ there is a universal function from $\kappa \times \kappa$ to n.*

3. *There is a symmetric, irreflexive function from $\kappa \times \kappa$ to 2 universal for all symmetric, irreflexive functions from $\kappa \times \kappa$ to 2*

4. *There is a universal graph on κ.*

Given Theorem 1.2 it is reasonable to focus attention only on symmetric functions from ω_1^2 to ω and this will be done from now on. Moreover, given that there are only two possible values for k when $\lambda = 2$, the validity of Remark 7.7 of [7] should now be clear and, of course, assuming the Continuum Hypothesis there is a Sierpiński universal function from $\mathbb{R}^2$ to $\mathbb{R}$, and hence there is also a weakly universal function from ω_1^2 to 2. Problem 7.8 of [7] asks if the existence of one type of universal function implies the existence of the others in general. To provide a negative answer to the question, it is therefore necessary to consider models where the Continuum Hypothesis fails. This provides an other path to the questions considered in [1], [2] and [3].

When $\kappa = \omega_1$ and $\lambda = 2$ it was shown in [1] and [2] that it is consistent with the failure of the Continuum Hypothesis that there is a symmetric universal function $U : \omega_1^2 \to 2$ which satisfies all three universality properties for symmetric functions. The methods used in the generalization of this result by Mekler to other theories in [3] provide examples of Sierpinski universal functions $U : \omega_1^2 \to \lambda$ for λ equal to 2, ω or ω_1. In the models of [1] and [2] the cardinal invariants $\mathfrak{b}$ and $\mathfrak{d}$ have the following values respectively: $\aleph_1$ and $\aleph_2$. One might, therefore ask, whether these values are needed for the existence of universal functions from $U : \omega_1^2 \to \lambda$ with the failure of the Continuum Hypothesis. This question becomes even more interesting in light of the positive results to be presented in Lemma 6.1 and Lemma 7.5 which rely on small values of $\mathfrak{b}$ and $\mathfrak{d}$ respectively. It will be shown in Corollary 3.10 that it is consistent with the existence of universal functions from $U : \omega_1^2 \to 2$ that $\mathfrak{b} = \mathfrak{d} = \aleph_2$ and in Corollary 4.14 that it is consistent with the existence universal functions from $U : \omega_1^2 \to 2$ that $\mathfrak{b} = \mathfrak{d} = \aleph_1$.

However, the main goal of this paper will be to answer Problem 5.10 of [7]. This is done in Corollary 6.2 which establishes that there is a Sierpiński universal function from ω_1^2 to 2 but no such function from ω_1^2 to ω. It is worth recalling that Theorem 5.9 of [7] asserts that if $2 \leq n < \omega$ then there is a Sierpiński universal function from ω_1^2 to 2 if and only if there is a Sierpiński universal function from ω_1^2 to n. The arguments presented here will also shed some light on Problem 7.8 of [7]. This and related questions are discussed in §7. Finally, if one is only interested in obtaining a model of set theory in which the Continuum Hypothesis fails, yet there is a universal graph of cardinality $\aleph_1$, then §2 presents an easier argument than the original of [1].

1.2. Notation and terminology

Since trees will play a central role in the following discussion, it may be worthwhile reviewing some notation and terminology, even though most of this is standard and almost all of the notation used will follow that of Sections 1.1.D and 7.3.D of [8]. By a tree T will be meant a subset $T \subseteq \omega^{<\omega} = \bigcup_{n\in\omega} \omega^n$ that is closed under initial segments — in other words, if $t \in T$ and $k \leq |t|$ then $t \restriction k \in T$. If T is a tree and $t \in T$ then $T[t]$ will denote the tree defined by

$$T[t] = \{s \in T \mid s \subseteq t \text{ or } t \subseteq s\}$$

and $\mathbf{succ}_T(t)$ will denote the set $\{s \in T \mid s \supseteq t \text{ and } |s| = |t| + 1\}$.

A tree T will be called *infinite splitting* if $|\mathbf{succ}_T(t)| \in \{1, \aleph_0\}$ for each $t \in T$. Define $\mathbf{split}(T) = \{t \in T \mid |\mathbf{succ}_T(t)| = \aleph_0\}$ and define

$$\mathbf{split}_n(T) = \{t \in \mathbf{split}(T) \mid |\{k \in |t| \mid t \restriction k \in \mathbf{split}(T)\}| = n\}.$$

If T is infinite splitting then let $\Psi_T : \omega^{<\omega} \to \mathbf{split}(T)$ be the unique bijection from $\omega^{<\omega}$ to $\mathbf{split}(T)$ preserving the lexicographic ordering. For $t \in \omega^{<\omega}$ let $T\langle t\rangle = T[\Psi_T(t)]$ — the reader is warned that this notation differs from [7]. Hence $\mathbf{stem}(T)$ can be defined to be $\Psi_T(\emptyset)$ for infinite splitting trees T.

Let $\{u_i\}_{i\in\omega}$ enumerate $\omega^{<\omega}$ in such a way that if $k < |u_i|$ then there is $j \in i$ such that $u_i \restriction k = u_j$. Then for infinite splitting trees T and S the ordering $\leq_n$ is defined by $T \leq_n S$ if $T \subseteq S$ and $\Psi_S(u_j) = \Psi_T(u_j)$ for all $j \in n$.

Finally, recall that Miller forcing, or rational perfect set forcing, is denoted by $\mathbf{PT}$ in [8] and consists of all infinite branching trees ordered by inclusion. Laver forcing, on the other hand, is denoted by $\mathbf{LT}$ in [8] and consists of all infinite branching trees such that $T \setminus \mathbf{split}(T)$ is finite, also ordered by inclusion. In the case of Laver forcing the notion of a *front* is useful: If $T \in \mathbf{LT}$ then $W \subseteq T$ is a front if it consists of incomparable elements of T and every maximal branch of T contains an element of W.

1.3. The universal graphs

The graphs that will be shown to be universal in the arguments to follow will all come from some initial model of set theory in which the Continuum Hypothesis holds.

Definition 1.3. Given any function $G : \omega_1^2 \to \omega$ and $\eta \in \omega_1$ define $G^\eta : \eta \to \omega$ by $G^\eta(\zeta) = G(\zeta, \eta)$ and then define $S^\eta(G) = \{G^\mu \restriction \eta\}_{\mu \geq \eta}$. A function $G : \omega_1^2 \to \omega$ such that $S^\eta(G)$ is everywhere non-meagre for each $\eta \geq \omega$ will be called *category saturated*.

If one does not wish to restrict to irreflexive functions then one can define

$$S^{\eta,k}(G) = \{G^\mu \restriction \eta \mid \mu \geq \eta \text{ and } G(\mu, \mu) = k\}$$

and then ask that $S^{\eta,k}(G)$ is everywhere non-meagre for each $\eta \geq \omega$ for each $k \in \omega$. Since this added complication currently provides no new insights, this work will focus on irreflexive functions only.

Definition 1.4. Let ν be an atomic probability measure on ω and let ν^η be the Fubini product of this measure on ω^η for any $\eta \in \omega_1$. A function $G : \omega_1^2 \to \omega$ such that $S^\eta(G)$ has outer measure 1 for each $\eta \geq \omega$ will be called *ν-saturated.* The notion defined here will not be needed in this full generality. For the purposes of this article a function $G : \omega_1^2 \to 2$ will be called *measure saturated* if it is ν-saturated where ν is the measure on 2 giving each point equal measure.

Lemma 1.5. *Assuming the Continuum Hypothesis, there is a symmetric function from ω_1^2 to ω that is category saturated and ν-saturated for every atomic probability measure ν.*

Indeed, using the Continuum Hypothesis it is easy to construct a symmetric function $G : \omega_1^2 \to \omega$ such that $S^\eta(G) = \omega^\eta$ for each $\eta \in \omega_1$. Note, however, that adding a real will destroy this stronger property; nevertheless, in certain generic extensions the weaker properties of being category saturated or ν-saturated may persist.

2. Category saturated graphs are universal after adding Miller reals

2.1. A P-ideal from a class of names

This section will describe an ideal that arises in generic extensions in which reals are added. It can easily, but inaccurately, be described as the collection of all sets in the extension that have a name that is a continuous function whose range is almost disjoint.

Definition 2.1. If $G \subseteq \mathbf{PT}$ is generic over V define $\mathcal{S}(\mathbf{PT})$ to be the set of all $S \in [\omega_1]^{\aleph_0}$ such that there is $T \in G$ and $\psi : T \to [\omega_1]^{<\aleph_0}$ such that

1. $\psi \in V$
2. $\psi(s) \cap \psi(t) = \emptyset$ unless $s = t$
3. $T \Vdash_{\mathbf{PT}}$ "$\dot{S} = \bigcup_{j=0}^\infty \psi(r_{\dot{G}} \restriction j)$" where $r_G : \omega \to \omega$ is the generic real obtained from the generic set G.

Functions satisfying (2) will be said to have *disjoint range.* Given any $\psi : T \to [\omega_1]^{<\aleph_0}$ let S_ψ be a $\mathbf{PT}$-name for $\bigcup_{j=0}^\infty \psi(r_G \restriction j)$ where $r_G = \bigcup_{T \in G} \mathbf{stem}(T)$ and G is the generic set.

Lemma 2.2. *If $S \in \mathcal{S}(\mathbf{PT})$ then there is $T \in G$ and $\psi : T \to [\omega_1]^{<\aleph_0}$ such that $\psi(t) = \emptyset$ unless $t \in \mathbf{split}(T)$ and $T \Vdash_{\mathbf{PT}}$ "$\dot{S} = S_\psi$".*

Proof. Given $S \in \mathcal{S}(\mathbf{PT})$ it suffices to show that for $T \in \mathbf{PT}$ there is $\bar{T} \subseteq T$ and $\psi : \bar{T} \to [\omega_1]^{<\aleph_0}$ such that $\psi(t) = \emptyset$ unless $t \in \mathbf{split}(\bar{T})$ and $\bar{T} \Vdash_{\mathbf{PT}}$ "$\dot{S} = S_\psi$". To this end use the definition of $\mathcal{S}(\mathbf{PT})$ to find T^*, $\hat{T}$ and and $\hat{\psi} : \hat{T} \to [\omega_1]^{<\aleph_0}$ such that $T^* \Vdash_{\mathbf{PT}}$ "$\dot{S} = S_{\hat{\psi}}$ and $\hat{T} \in \dot{G}$". Observe that $\bar{T} = T^* \cap \hat{T} \in \mathbf{PT}$ because otherwise it would follow that $T^* \Vdash_{\mathbf{PT}}$ "$\hat{T} \notin \dot{G}$".

Then define $\psi : \bar{T} \to [\omega_1]^{<\aleph_0}$ by

$$\psi(t) = \begin{cases} \bigcup \left\{ \hat{\psi}(t \restriction k) \mid (\forall j \in [k, |t|))\ t \restriction j \notin \mathbf{split}(\bar{T}) \right\} & \text{if } t \in \mathbf{split}(\bar{T}) \\ \emptyset & \text{otherwise.} \end{cases}$$

It should be obvious that $\psi : \bar{T} \to [\omega_1]^{<\aleph_0}$ has disjoint range and that $\bar{T} \Vdash_{\mathbf{PT}}$ "$S_\psi = S_{\hat{\psi}} = \dot{S}$".

Lemma 2.3. *Let $G \subseteq \mathbf{PT}$ be generic over V and $S \subseteq \omega_1$ in $V[G]$. Then the following are equivalent:*

1. *$S \in \mathcal{S}(\mathbf{PT})$*
2. *for all $F : \omega \times \omega \to \omega_1$ in V such that the mapping $j \mapsto F(n, j)$ is one-to-one for all $n \in \omega$, there is $g : \omega \to \omega$ in V such that for all $n \in \omega$ there is $k \le g(n)$ such that $F(n, k) \notin S$*
3. *for all $F : \omega \times \omega \to \omega_1$ in V such that the mapping $j \mapsto F(n, j)$ is one-to-one for all $n \in \omega$, there is a one-to-one function $g : \omega \times \omega \to \omega$ in V such that $F(n, g(n, m)) \notin S$ for all $(n, m) \in \omega \times \omega$.*

In particular, $\mathcal{S}(\mathbf{PT})$ is closed under subsets and finite unions. Moroever if $S \in \mathcal{S}(\mathbf{PT})$ then S contains no infinite set from V.

Proof. To see that (1) implies (3) suppose that $S \in \mathcal{S}(\mathbf{PT})$. Using Lemma 2.2 find

$$\psi : \mathbf{split}(T) \to [\omega_1]^{<\aleph_0}$$

such that $T \Vdash_{\mathbf{PT}}$ "$\dot{S} = S_\psi$" and $\psi(t) = \emptyset$ unless $t \in \mathbf{split}(T)$. Let $F : \omega \times \omega \to \omega_1$ belong to V such that the mapping $j \mapsto F(n, j)$ is one-to-one for all $n \in \omega$. Let $\{(i_n, j_n)\}_{n\in\omega}$ enumerate $\omega \times \omega$ and construct trees T_n and distinct integers $g(i_n, j_n)$ such that

1. $T_0 = T$
2. $T_{n+1} \le_n T_n$
3. $F(\ell, g(i_\ell, j_\ell)) \notin \bigcup_{i\in n} \bigcup_{j\le|\Psi_{T_n}(u_i)|} \psi(\Psi_{T_n}(u_i) \restriction j)$ for all $\ell \in n$.

To construct T_{n+1} first choose $g(i_n, j_n)$ such that $g(i_n, j_n) \ne g(i_\ell, j_\ell)$ for $\ell \in n$ and such that

$$F(i_n, g(i_n, j_n)) \notin \bigcup_{i\in n} \bigcup_{j\le|\Psi_{T_n}(u_i)|} \psi(\Psi_{T_n}(u_i) \restriction j).$$

Then note that for each $k \le n$ there is at most one $s_k \in T_n$ such that $F(i_k, g(i_k, j_k)) \in \psi(s_k)$ and none of these s_k is below any $\Psi_{T_n}(u_\ell)$ for $\ell \in n$. It is therefore easy to find $T_{n+1} \subseteq T_n$ such that $\Psi_{T_{n+1}}(u_n) \ne s_i$ for all $i \le n$ and, hence, the induction hypotheses are satisfied. Let $T^* = \bigcap_n T_n$ and observe that the function $g : \omega \times \omega \to \omega$ defined by the induction belongs to V and that

$$T^* \Vdash_{\mathbf{PT}} \text{"}(\forall n \in \omega)\ F(n, g(i_n, j_n)) \notin \dot{S}\text{"}.$$

To see that (2) implies (1) suppose that S satisfies (2). Let $S = \{\sigma_j\}_{j\in\omega}$ be an enumeration of S and let $T \in G$ be such that there is $\psi : \mathbf{split}_n(T) \to \omega_1$ such that $T_t \Vdash_{\mathbf{PT}}$ "$\dot{\sigma_n} = \psi(t)$" for each $t \in \mathbf{split}_n(T)$. Now let B be the set of $t \in \mathbf{split}(T)$ such that there is a family $\{w^t_i\}_{i\in\omega} \subseteq \mathbf{split}(T)$ such that

1. $w^t_i \supseteq t$
2. $w_i(|t|) \ne w_j(|t|)$ unless $i = j$
3. there is $\theta_n \in \prod_{i=n}^\infty (|t|, |w^t_i|)$ and ρ_n for each $n \in \omega$ such that if $n \le i$ then $\psi(w^t_i \restriction \theta_n(i)) = \rho_n$
4. $\rho_n \ne \rho_m$ unless $n = m$.

It will first be shown that B cannot be dense in T.

For if B were dense in T then it would be possible to find $\bar{T} \subseteq T$ such that $\bar{T} \in \mathbf{PT}$, $\mathbf{split}(\bar{T}) \subseteq B$ and if $t \in \mathbf{split}_m(\bar{T})$ and $\{w^t_i\}_{i\in\omega}$ enumerates $\left\{s \in \mathbf{split}_{m+1}(\bar{T}) \mid s \supseteq t\right\}$ then there are $\theta^t_n \in \prod_{i=n}^\infty (|t|, |w^t_i|)$ and ρ^t_n for each $n \in \omega$ witnessing that $t \in B$. Let $\{t_i\}_{i\in\omega}$ enumerate $\mathbf{split}(\bar{T})$ and define $F : \omega \times \omega \to \omega_1$ by $F(m, n) = \rho^{t_m}_n$.

Now given any $T^* \subseteq \bar{T}$ such that $T^* \in \mathbf{PT}$ and any $g \in \omega^\omega \cap V$ it follows that $\mathbf{stem}(T^*) = t_m$ for some m. There is then some $n \in \omega$ and some t^* such that $t^* \supseteq w^{t_m}_n \supseteq t_m$ and $n > g(m)$. Then $T^*_{t^*} \Vdash_{\mathbf{PT}}$ "$\{\rho^{t_m}_i\}_{i=0}^{g(m)+1} \subseteq \dot{S}$" yielding a contradiction to the hypothesis on S.

Hence it will be assumed that $B = \emptyset$. Construct a sequence $T = T_0 \ge_0 T_1 \ge_1 T_2 \ldots$ such that for each $k \in n$ there is a family $\{w^k_i\}_{i\in\omega}$ satisfying:

1. $w^k_i \supseteq \Psi_{T_n}(u_k)$

2. $w_i^k(|\Psi_{T_n}(u_k)|) \neq w_m^k(|\Psi_{T_n}(u_k)|)$ unless $i = m$

3. there is some $J_k \in \omega$ and $\theta_j \in \prod_{i=j}^{\infty}(|\Psi_{T_n}(u_k)|, |w_i^k|)$ and distinct ρ_j^k for each $j \in J_k$ such that if $j \leq i$ then $\psi(w_i^k \restriction \theta_j(i)) = \rho_j^k$ and, moreover, the set $\{\rho_j^k\}_{j\in J_k}$ is maximal with this property.

Completing this induction is easy using the fact that $T_n \cap B = \emptyset$. Let $T^* = \bigcap_n T_n$ and let $\bar\psi(\Psi_{T^*}(u_k)) = \{\rho_j^k\}_{j\in J_k}$.

Next observe that there is $T^{**} \subseteq T^*$ such that $\mathbf{split}(T^{**}) = \mathbf{split}(T^*) \cap T^{**}$ and

$$(\forall s \in T_n \setminus \{\Psi_{T_n}(u_k) \mid k \in n\})\ \psi(s) \notin \bigcup_{k\in n} \bar\psi(\Psi_{T_n}(u_k)) \tag{2.1}$$

$$(\forall k < m < n)\ \bar\psi(\Psi_{T_n}(u_k)) \cap \bar\psi(\Psi_{T_n}(u_m)) = \emptyset \tag{2.2}$$

To see this construct a sequence $T = T_0 \geq_0 T_1 \geq_1 T_2 \ldots$ such that $\bar\psi(\Psi_{T_n}(u_i)) \cap \bar\psi(\Psi_{T_n}(u_j)) = \emptyset$ for $i < j < n$. Given T_n, let j be such that u_j is maximal from the set $\{u_i \mid i < n \text{ and } u_i \subseteq u_n\}$. There cannot be infinitely many $s \in \mathbf{succ}_{T_n}(\Psi_{T_n}(u_j))$ such that there is $\bar s \in T_n[s]$ such that $\psi(\bar s) \in \bigcup_{k\in n} \bar\psi(\Psi_{T_n}(u_k)) \neq \emptyset$ because this would violate the maximality of $\bar\psi(\Psi_{T_n}(u_j))$. This makes it easy to find T_{n+1}. Then $\bar\psi : T^{**} \to [\omega_1]^{<\aleph_0}$ witness that $S \in \mathcal{S}(\mathbf{PT})$.

To see that (3) implies (2) let $g(n) = g(n, 0) + 1$.

The fact that $\mathcal{S}(\mathbf{PT})$ is closed under subsets follows from (3). To see that $\mathcal{S}(\mathbf{PT})$ is closed under finite unions, let S_0 and S_1 belong to $\mathcal{S}(\mathbf{PT})$. Given $F : \omega \times \omega \to \omega_1$ let $g_0 : \omega \times \omega \to \omega$ witness (2) for S_0. Then, using $F \circ g_0$ in the place of F let $g_1 : \omega \times \omega \to \omega$ witness (2) for S_1. This also witnesses (2) for $S_0 \cup S_1$ for the given function F.

To see the final claim le $X \in [\omega_1]^{\aleph_0}$ belong to V and let $S \in \mathcal{S}(\mathbf{PT})$. Then let $F : \omega \times \omega \to \omega_1$ be such that $\{F(i, j)\}_{j\in\omega}$ enumerates X on each $i \in \omega$. It follows from (3) that $X \not\subseteq S$.

It will be useful to observe that it is possible to isolate part of the proof of Lemma 2.3 to show the following.

Corollary 2.4. *If $T \in \mathbf{PT}$ and $\psi_i : T \to [\omega_1]^{<\aleph_0}$ have disjoint range for $i \in k$ then there is $T^* \subseteq T$ such that $\mathbf{split}(T^*) = \mathbf{split}(T) \cap T^*$ and the function ψ defined by $\psi(t) = \bigcup_{i\in k} \psi_i(t)$ has disjoint range.*

Proof. The key point is that at each stage of a fusion argument only finitely many nodes of the tree need to eliminated.

Question 2.5. If G is $\mathbf{PT}$ generic over V does it follow that $\mathcal{S}(\mathbf{PT})$ equal to the

$$\left\{X \in [\omega_1]^{\aleph_0} \;\middle|\; \mathcal{P}(X) \cap V = [X]^{<\aleph_0}\right\}$$

in $V[G]$?

Lemma 2.6. *If $\psi_i : T_i \to [\omega_1]^{<\aleph_0}$ have disjoint range for $i \in \omega$ then there are $\bar T_i \subseteq T_i$ such that*

$$\mathbf{split}(\bar T_i) = \mathbf{split}(T_i) \cap \bar T_i \tag{2.3}$$

$$(\forall i < j < \omega)(\forall t \in \bar T_i)(\forall s \in \bar T_j)\ \text{ if } t \supsetneq \mathbf{stem}(T_i) \text{ and } s \supsetneq \mathbf{stem}(T_j) \text{ then } \psi_j(s) \cap \psi_i(t) = \emptyset \tag{2.4}$$

Proof. This is a routine fusion argument. Using Lemma 2.2 it may be assumed that $\psi_i(t) = \emptyset$ unless $t \in \mathbf{split}(T_i)$. The key point is that given $n \in \omega$ and $T_{n,i} \subseteq T_i$ for $i \in \omega$ there are $T_{n+1,j}$ for $j \in \omega$ such that

- $T_{n+1,j} \leq_n T_{n,j}$ for $j \in n$
- $T_{n+1,j} \subseteq T_{n,j}$ for $j \in \omega$
- $\mathbf{split}(T_{n+1,j}) = \mathbf{split}(T_{n,j}) \cap T_{n+1,j}$ for $j \in \omega$
- $T_{n,j}\langle u_i\rangle = T[\Psi_{T_{n,j}}(u_i)]$ for $j \leq n$ and $i \in n$
- $\psi_i(\Psi_{T_{n,i}}(u_m)) \cap \psi_j(\Psi_{T_{n,j}}(u_k)) = \emptyset$ if $i < j \leq n$ and $k, m \in n$.

To find $\Psi_{T_{n+1,j}}(u_n)$ for $j \leq n$ use the fact that each ψ_i has disjoint range and each $T_{n,j}$ is infinite branching to eliminate the finitely many nodes violating this requirement — only $\{\mathbf{stem}(T_i)\}_{i\in\omega}$ cannot be omitted.

Lemma 2.7. *If S is $\mathbf{PT}$-name such that $T \Vdash_{\mathbf{PT}}$ "$S \in \mathcal{S}(\mathbf{PT})$" then there is $\bar T \subseteq T$ and $\psi : \bar T \to [\omega_1]^{<\aleph_0}$ such that*

- $\mathbf{stem}(\bar{T}) = \mathbf{stem}(T) = t^*$
- *ψ has disjoint range*
- *$\psi(t^* \restriction j) = \emptyset$ for $j \leq |t^*|$*
- *$\bar{T} \Vdash_{\mathbf{PT}}$ "$S \equiv^* S_\psi$".*

PROOF. For each $t \in \mathbf{succ}_T(t^*)$ use Lemma 2.2 to find $T_t \leq T[t]$ and ψ_t such that

- $\psi_t : T_t \to [\omega_1]^{<\aleph_0}$ has disjoint range
- $\psi_t(t \restriction j) = \emptyset$ for $j \leq |t|$
- $T_t \Vdash_{\mathbf{PT}}$ "$S \equiv^* S_{\psi_t}$".

Now apply Lemma 2.6 to the family $\{\psi_t\}_{t \in \mathbf{succ}_T(t^*)}$ to find $\tilde{T}_t \subseteq T_t$ such that ψ defined by

$$\psi(s) = \begin{cases} \psi_t(s) & \text{if } (\exists t \in \mathbf{succ}_T(t^*))\ s \supsetneq t \text{ and } s \in \tilde{T}_t \\ \emptyset & \text{if } (\exists t \in \mathbf{succ}_T(t^*))\ s \subseteq t \end{cases}$$

has disjoint range. Let $\bar{T} = \bigcup_{t \in \mathbf{succ}_T(t^*)} \tilde{T}_t$. It is immediate that ψ and $\bar{T}$ satisfy the lemma.

Lemma 2.8. *If G is $\mathbf{PT}$ generic over V then $\mathcal{S}(\mathbf{PT})$ is a P-ideal in $V[G]$.*

PROOF. Suppose that $\{S_n\}_{n \in \omega}$ are $\mathbf{PT}$-names such that $T \Vdash_{\mathbf{PT}}$ "$S_n \in \mathcal{S}(\mathbf{PT})$" for each n. If it is possible to construct T_n and ψ_n such that

1. $T_0 = T$
2. $T_{n+1} \leq_n T_n$
3. $\psi_n : T_n \to [\omega_1]^{<\aleph_0}$ has disjoint range
4. $\psi_n(\Psi_{T_n}(u_i) \restriction j) = \psi_{n+1}(\Psi_{T_n}(u_i) \restriction j)$ if $i \in n$ and $j \leq |\Psi_{T_n}(u_i) \restriction j|$
5. if $t \in T_{n+1}$ then $\psi_n(t) \subseteq \psi_{n+1}(t)$
6. $(\forall i \leq n)\ T_n \Vdash_{\mathbf{PT}}$ "$S_i \subseteq^* S_{\psi_n}$".

Then let $\bar{T} = \bigcap_{k \in \omega} T_k$ and let ψ be defined by $\psi(t) = \bigcup_k \psi_k(t)$ noting that this is finite by (4). Then

$$\bar{T} \Vdash_{\mathbf{PT}} \text{“}S_\psi \in \mathcal{S}(\mathbf{PT}) \text{ and } (\forall j)\ S_\psi \supseteq^* S_j\text{”}.$$

To complete the construction suppose that T_n and ψ_n are given. For $i \in n$ let

$$U_i = \{t \in T\langle u_i \rangle \mid (\forall j \in n) \text{ if } u_j \subseteq t \text{ then } u_j \subseteq u_i\}.$$

Use Lemma 2.7 to find $T_i^* \subseteq U_i$ and $\psi_i^* : T_i^* \to [\omega_1]^{<\aleph_0}$ such that

- $\mathbf{stem}(T_i^*) = \Psi_{T_n}(u_i)$
- ψ_i^* has disjoint range
- $\psi_i^*(\Psi_{T_n}(u_i) \restriction j) = \emptyset$ for $j \leq |\Psi_{T_n}(u_i)|$
- $T_i^* \Vdash_{\mathbf{PT}}$ "$S_n \equiv^* S_{\psi_i^*}$".

Use Corollary 2.4 for each $i \in n$ to find $T_i^{**} \subseteq T_i^*$ such that $\mathbf{stem}(T_i^{**}) = \Psi_{T_n}(u_i)$ and the function ψ_i^{**} defined by $\psi_i^{**}(s) = \psi_i^*(s) \cup \psi_n(s)$ has disjoint range. Then apply Lemma 2.6 to find $\tilde{T}_i \subseteq T_i^{**}$ such that $\mathbf{stem}(\tilde{T}_i) = \Psi_{T_n}(u_i)$ and

$$(\forall i < j < n)(\forall t \in \tilde{T}_i)(\forall s \in \tilde{T}_j) \text{ if } t \supsetneq \Psi_{T_n}(u_i) \text{ and } s \supsetneq \Psi_{T_n}(u_j) \text{ then } \psi_j^{**}(s) \cap \psi_i^{**}(t) = \emptyset.$$

Since this only requires a finite amount of further pruning, it may be assumed that

$$\psi_i^{**}(t) \cap \left(\bigcup_{j \in n} \left(\bigcup_{k \leq |\Psi_{T_n}(u_j)|} \psi_n \left(\Psi_{T_n}(u_j) \restriction k \right) \right) \right) = \emptyset$$

for each $i \in n$ and $t \in \tilde{T}_i$. Now let $T_{n+1} = \bigcup_{i \in n} \tilde{T}_i$ and define ψ_{n+1} by

$$\psi_{n+1}(t) = \begin{cases} \psi_n(t) & \text{if } (\exists i \in n)\ t \subseteq \Psi_{T_n}(u_i) \\ \psi_n(t) \cup \psi_i^{**}(t) & \text{if } (\exists i \in n)\ t \supsetneq \Psi_{T_n}(u_i) \end{cases}$$

and check that this satisifies the induction hyptheses.

Lemma 2.9. *If G is* **PT** *generic over V and $S \in \mathcal{S}(\mathbf{PT})$ and $f : S \to 2$ is a function in $V[G]$ then there is $T \in G$ and ψ defined on T with disjoint range and f^* such that*

1. $T \Vdash_{\mathbf{PT}}$ "$\dot{S} = S_\psi$"
2. $f^* : \bigcup_{t \in T} \psi(t) \to 2$
3. *if $t \in T$ then $T[t] \Vdash_{\mathbf{PT}}$ "$\dot{f} \restriction \psi(t) = f^* \restriction \psi(t)$" and, hence, $T \Vdash_{\mathbf{PT}}$ "$\dot{f} = f^* \restriction S$".*

Proof. It suffices to show that the set of $T \in \mathbf{PT}$ satisfying Conditions (1) to (3) is dense. Given $T \in \mathbf{PT}$ use Lemma 2.2 to find $\bar{T} \subseteq T$ and ψ such that $\bar{T} \Vdash_{\mathbf{PT}}$ "$\dot{S} = S_\psi$" and such that $\psi(t) = \emptyset$ unless $t \in \mathbf{split}(\bar{T})$. Now construct T_n and f_n^* such that:

- $T_0 = \bar{T}$
- $T_{n+1} \leq_n T_n$
- the domain of f_n^* is $\bigcup_{j \in n} \psi(\Psi_{T_n}(u_j))$
- if $j \in n$ then $T_n \langle u_j \rangle \Vdash_{\mathbf{PT}}$ "$f_n^* \restriction \psi(\Psi_{T_n}(u_j)) \subseteq \dot{f}$".

Then if $T_\omega = \bigcap_{n \in \omega} T_n$ it is clear that $\psi \restriction T_\omega$ and $f^* = \bigcup_{n \in \omega} f_n^*$ witness that T_ω satisfies Conditions (1) to (3) of the lemma.

To complete the induction it suffices to note that $T_n \langle u_n \rangle \Vdash_{\mathbf{PT}}$ "$\psi(\Psi_{T_n}(u_n)) \subseteq \dot{S}$" and hence for each $s \in \mathbf{succ}_{T_n}(\Psi_{T_n}(u_n))$ there is an extension $T^s \subseteq T_n[s]$ and f_s such that $T^s \Vdash_{\mathbf{PT}}$ "$\dot{f} \restriction \psi(\Psi_{T_n}(u_n)) = f_s$". Now choose T_{n+1} such that $T_{n+1} \leq_n T_n$ and there is $\bar{f} : \psi(\Psi_{T_n}(u_n)) \to 2$ such that $f_s = \bar{f}$ for all $s \in \mathbf{succ}_{T_{n+1}}(\Psi_{T_n}(u_n))$. Let $f_{n+1}^* = f_n^* \cup \bar{f}$.

Lemma 2.10. *If G is* **PT** *generic over V and $S \in \mathcal{S}(\mathbf{PT})$, $S \subseteq \xi \in \omega_1$, $f : S \to 2$ is a function in $V[G]$ and $Z \subseteq 2^\xi$ is nowhere meagre, then there is $z \in Z$ such that $f \subseteq z$.*

Proof. Given f find T, ψ and f^* satisfying Conditions (1) to (3) of Lemma 2.9. Using Lemma 2.2 it may be assumed that $\psi(t) = \emptyset$ if $t \notin \mathbf{split}(T)$. Let $\bar{f} = f^* \restriction \psi(\mathbf{stem}(T))$ and let $\mathcal{O}$ be the open set $\{h \in 2^\xi \mid h \supseteq \bar{f}\}$. Then $Z \cap \mathcal{O}$ is not meagre in $\mathcal{O}$. Note that

$$\mathcal{O} \cap \left\{ h \in 2^\xi \;\middle|\; (\forall k \in \omega)(\forall t \in \mathbf{split}_k(T))\ |\left\{ s \in \mathbf{split}_{k+1}(T) \mid s \supseteq t \text{ and } f^* \restriction \psi(s) \subseteq h \right\}| = \aleph_0 \right\}$$

is a dense G_δ in $\mathcal{O}$ and hence there is some $z \in Z$ such that $\bar{f} \subseteq z$ and

$$(\forall k \in \omega)(\forall t \in \mathbf{split}_k(T))\ |\left\{ s \in \mathbf{split}_{k+1}(T) \mid s \supseteq t \text{ and } f^* \restriction \psi(s) \subseteq h \right\}| = \aleph_0$$

It follows that there is $T^* \subseteq T$ such that $T^* \in \mathbf{PT}$ and such that if $t \in \mathbf{split}(T^*)$ then $f^* \restriction \psi(t) \subseteq z$ and hence $f^* \subseteq z$. It follows that $T^* \Vdash_{\mathbf{PT}}$ "$\dot{f} \subseteq z$".

2.2. Applying the P-ideal dichotomy

This section will use some results from [9]. (The strengthened results of [10] will not be needed here). Recall that if $\mathcal{I}$ is an ideal on a set X then $Y \subseteq X$ is said to be orthogonal to $\mathcal{I}$ if $[Y]^{\aleph_0} \cap \mathcal{I} = \emptyset$.

Lemma 2.11. *If G is* **PT** *generic over V then no uncountable subset of ω_1 is orthogonal to $\mathcal{S}(\mathbf{PT})$ in $V[G]$.*

Proof. Suppose that Z is a **PT**-name such that $T \Vdash_{\mathbf{PT}}$ "$Z \in [\omega_1]^{\aleph_1}$". It suffices to construct a sequence of conditions $T_n \in \mathbf{PT}$ and ordinals ζ_n such that:

- $T_0 = T$,
- $T_{n+1} \leq_n T_n$ for each n
- $T_n\langle u_j\rangle \Vdash_{\mathbf{PT}}$ "$\zeta_j \in Z$" for each $j \in n$
- the mapping $j \mapsto \zeta_j$ is one-to-one.

To carry out the construction let T_n be given and let $\eta = \max_{j\in n} \zeta_{u_j}$. Find $T^* \subseteq T_n\langle u_n\rangle$ and $\zeta_n > \eta$ such that $T^* \Vdash_{\mathbf{PT}}$ "$\zeta_n \in Z$". Let $T_{n+1} = (T_n \setminus T_n\langle u_n\rangle) \cup T^*$.

The applications of the P-ideal dichotomy in [9] and [10] all rely on simply finding an uncountable set all of whose countable subsets belong to a given ideal. The arguments to be presented here rely on a stronger version of the axiom, but one which is, nevertheless, true in the model constructed in [9]. The following theorem is implicit in Lemma 3.1 of [9]; the following argument simply verifies this assertion.

Theorem 2.12 (Abraham & Todorcevic). *If $\mathcal{I}$ is a P-ideal on ω_1 then there is a partial order $\mathbb{P}_{\mathcal{I}}$, that adds no reals, even when iterated with countable support, such that $\mathbb{P}_{\mathcal{I}}$ adds a set $Z \subseteq \omega_1$ such that for any $W \subseteq \omega_1$ which is not the union of countably many sets orthogonal to $\mathcal{I}$*

$$1 \Vdash_{\mathbb{P}_{\mathcal{I}}} \text{“}\dot{Z} \cap W \neq \emptyset\text{”} \tag{2.5}$$

$$1 \Vdash_{\mathbb{P}_{\mathcal{I}}} \text{“}(\forall \eta \in \omega_1)\ \dot{Z} \cap \eta \in \mathcal{I}\text{”}. \tag{2.6}$$

Proof. The proof is the same as that in [9]. To begin, given a non-principal P-ideal $\mathcal{I}$ on ω_1, fix $A_\xi \subseteq \xi$ for each $\xi \in \omega_1$ such that if $\xi \in \eta$ then $A_\xi \subseteq^* A_\eta$ and such that every member of $\mathcal{I}$ is almost included in some A_ξ. The partial order $\mathbb{P}_{\mathcal{I}}$ is defined to consist of pairs $p = (x_p, \mathfrak{X}_p)$ such that:

- $x_p \in \mathcal{I}$ (The reader should not be confused by the claim in [9] that x_p can be any countable subset of ω_1.)
- $|\mathfrak{X}_p| \leq \aleph_0$
- $\mathfrak{X}_p \subseteq [\omega_1]^{\aleph_1}$.

The ordering on $\mathbb{P}_{\mathcal{I}}$ is defined by defining $p \leq q$ if:

- $x_p \cap \sup(x_q) = x_q$
- $\mathfrak{X}_p \supseteq \mathfrak{X}_q$
- $\{\xi \in X \mid x_p \setminus x_q \subseteq A_\xi\} \in \mathfrak{X}_p$ for every $X \in \mathfrak{X}_q$.

Lemma 3.1 of [9] establishes that if ω_1 cannot be decomposed into countably many sets orthogonal to $\mathcal{I}$, then for each $\gamma \in \omega_1$ the set of $p \in \mathbb{P}_{\mathcal{I}}$ such that $x_p \setminus \gamma \neq \emptyset$ is a dense subset of $\mathbb{P}_{\mathcal{I}}$. It will now be verified that the same argument establishes that if $W \subseteq \omega_1$ is not the union of countably many sets orthogonal to $\mathcal{I}$ then $D(W) = \{p \in \mathbb{P}_{\mathcal{I}} \mid x_p \cap W \neq \emptyset\}$ is a dense subset of $\mathbb{P}_{\mathcal{I}}$.

To see this suppose that there is no member of $D(W)$ extending p. Then for each $\mu \in W$ there is some $X \in \mathfrak{X}_p$ such that

$$X(\mu) = \{\xi \in X \mid \mu \in A_\xi\}$$

is countable. For $X \in \mathfrak{X}_p$ let

$$B(X) = \{\mu \in W \mid |X(\mu)| = \aleph_0\}$$

and note that the hypothesis on W implies that $\bigcup_{X\in\mathfrak{X}_p} B(X) \supseteq W$. It therefore suffices to show that each $B(X)$ is orthogonal to $\mathcal{I}$. To see that this is so, suppose that b is an infinite subset of $B(X)$ and that $b \in \mathcal{I}$. Then $\{\xi \in X \mid b \subseteq^* A_\xi\}$ is clearly a co-countable subset of X because of the cofinality of $\{A_\xi\}_{\xi\in\omega_1}$. Moreover

$$\{\xi \in X \mid b \subseteq^* A_\xi\} = \bigcup_{F\in[b]^{<\aleph_0}} \{\xi \in X \mid (b\setminus F) \subseteq A_\xi\}$$

and hence there is some $F \in [b]^{<\aleph_0}$ such that $\{\xi \in X \mid (b \setminus F) \subseteq A_\xi\}$ is uncountable. Then if $\mu \in b\setminus F$ this contradicts that $X(\mu) \supseteq \{\xi \in X \mid (b\setminus F) \subseteq A_\xi\}$ and $|X(\mu)| = \aleph_0$ because $\mu \in B(X)$.

Theorem 2.13. *Let V be a model of set theory and suppose that $U : \omega_1^2 \to 2$ is a symmetric, category saturated function in V and that $G \subseteq \mathbf{PT}$ is generic over V. In $V[G]$ let $H \subseteq \mathbb{P}_{\mathcal{S}(\mathbf{PT})}$ be generic over $V[G]$. Then in $V[G][H]$ the function U is universal.*

Proof. Using Theorem 2.12 in $V[G]$ there is $R \subseteq \omega_1$ such that $[R]^{\aleph_0} \subseteq \mathcal{S}(\mathbf{PT})$ and $R \cap Y \neq \emptyset$ for each uncountable $Y \in V[G]$. Given $W : \omega_1^2 \to 2$ which is symmetric, construct by induction embeddings $e_\eta : \eta \to R$ of $W \restriction \eta^2$ into U such that $e_\eta \subseteq e_\zeta$ if $\eta \leq \zeta$.

Since limit stages of the induction are trivial, it suffices to show that given e_η there is $e_{\eta+1}$ as required. Let S be the range of e_η and suppose that $S \subseteq \xi$. Then $S \in [R]^{\aleph_0} \subseteq \mathcal{S}(\mathbf{PT})$. Let $f : S \to 2$ be defined by $f(\sigma) = W(e_\eta^{-1}(\sigma), \eta)$ and note that $f \in V[G]$ since $V[G]$ and $V[G][H]$ have the same reals. Recall that $\mathbf{PT}$ preserves non-meagre sets by Theorems 6.3.20 and 7.3.46 in [8]. By Lemma 2.10 it therefore follows that, using the notation of Definition 1.3, $Y = \{\gamma \in \omega_1 \mid f \subseteq U^\gamma\}$ is an uncountable set in $V[G]$. By Theorem 2.12 it is possible to find $\gamma \in R\setminus\xi$ such that $f \subseteq U^\gamma$ and, hence, $W(e_\eta^{-1}(\sigma), \eta) = f(\sigma) = U(\sigma,\gamma)$ for all $\sigma \in \mu$. Let $e_{\eta+1} = e_\eta \cup \{(\eta,\gamma)\}$.

Corollary 2.14. *It is consistent with $\mathfrak{b} = \aleph_1$ and $\mathfrak{d} = \aleph_2$ that there is a universal graph on ω_1.*

Proof. The required model is the one obtained by starting with a model of the Continuum Hypothesis and iterating, with countable support, ω_2 Miller reals at even coordinates and forcing with $\mathbb{P}_{\mathcal{S}(\mathbf{PT})}$ at odd coordinates. Any category saturated graph in the initial model — and, in particular, any saturated graph — will be universal in the final extension. To see this, begin by observing that by Theorem 7.3.46 of [8] it follows that $\mathbf{PT}$ preserves $\sqsubseteq^{\mathbf{Cohen}}$ as defined in Definition 6.3.15 of [8]. Since $\mathbb{P}_{\mathcal{S}(\mathbf{PT})}$ is proper and adds no new reals it is immediate that it also preserves $\sqsubseteq^{\mathbf{Cohen}}$. By Theorem 6.3.20 of [8] it follows that the entire countable support iteration preserves non-meagre sets and, hence, any category saturated graph in the initial model remains category saturated.

To see that all of these graphs are universal use Lemma 3.4 and Lemma 3.6 of [9] to conclude that each partial order in the ω_2 length iteration is proper and has the $\aleph_2$-pic of Definition 2.1 on page 409 of [11]. By Lemma 2.4 on page 410 of [11] it follows that the iteration has the $\aleph_2$ chain condition and, hence, that any graph on ω_1 appears at some stage. It is then routine to apply Theorem 2.13.

It may be worth observing that Theorem 2.13 actually yields that if V is any model of set theory in which there is a category saturated graph and if G is $\mathbf{PT}$ generic over V and H is $\mathbb{P}_{\mathcal{S}(\mathbf{PT})}$ generic over $V[G]$ then already in the model $V[G][H]$ the category saturated graph is universal. So if there were a model of $2^{\aleph_0} > \aleph_1$ with a category saturated graph such that forcing with $\mathbf{PT} * \mathbb{P}_{\mathcal{S}(\mathbf{PT})}$ does not collapse the continuum, then this would yield an even simpler method for obtaining a universal graph with the failure of the continuum hypothesis. But even the following question seems to be open.

Question 2.15. Is there a model of set theory in which there is a non-meagre set of cardinality less than $2^{\aleph_0}$ such that forcing with $\mathbf{PT}$ over this model does not collapse the continuum?

3. Measure saturated graphs are universal after adding Laver reals

Definition 3.1. If $G \subseteq \mathbf{LT}$ is generic over V define $\mathcal{S}(\mathbf{LT})$ to be the set of all $S \subseteq \omega_1$ such that there is $T \in G$ and $\psi : T \to [\omega_1]^{<\aleph_0}$ such that

1. $\psi \in V$
2. if $s \neq t$ then $\psi(s) \cap \psi(t) = \emptyset$

3. for each $t \in T$ there is $K \in \omega$ such that $|\psi(s)| < K$ for all $s \in \mathbf{succ}_T(t)$.

4. $T \Vdash_{\mathbf{LT}}$ "$\dot{S} = S_\psi$"

recalling that S_ψ is defined in Definition 2.1. Using Lemma 2.2 it can and will be assumed that $\psi(t) = \emptyset$ if $t \subsetneq \mathbf{stem}(T)$.

Notation 3.2. For a tree $W \subseteq \omega^{<\omega}$ let $\max(W)$ denote the maximal elements of W.

Lemma 3.3. *If S is an $\mathbf{LT}$-name such that $T \Vdash_{\mathbf{LT}}$ "$S \in \mathcal{S}(\mathbf{LT})$" then there is $\bar{T} \subseteq T$ and $\psi : \bar{T} \to [\omega_1]^{<\aleph_0}$ such that*

- $\mathbf{stem}(T) = \mathbf{stem}(\bar{T})$
- *ψ has disjoint range*
- $\bar{T} \Vdash_{\mathbf{LT}}$ "*$S \equiv^* S_\psi$*".

Proof. For each $w \in \mathbf{succ}_T(\mathbf{stem}(T))$ let $T_w \subseteq T[w]$ be such that $\mathbf{stem}(T_w) = w$ and there is a well founded tree $W_w \subseteq T_w$ such that $\max(W_w)$ is a front and for each $t \in \max(W_w)$ there is $\psi_t : T_w[t] \to [\omega_1]^{<\aleph_0}$ with disjoint range such that

$$T_w[t] \Vdash_{\mathbf{LT}} \text{“}S_{\psi_t} = S\text{”}.$$

The existence of W_w follows from a standard rank argument using the fact that for every $U \in \mathbf{LT}$ such that $U \Vdash_{\mathbf{LT}}$ "$S \in \mathcal{S}(\mathbf{LT})$" there is $\bar{U} \subseteq U$ such that $\bar{U} \Vdash_{\mathbf{LT}}$ "$S = S_\psi$" for some $\psi : \bar{U} \to [\omega_1]^{<\aleph_0}$ in V with disjoint range.

Note that Lemma 2.6 remains true for Laver forcing because of Clause 2.3 and apply it to the family

$$\{\psi_t \restriction T_w[t] \mid w \in \mathbf{succ}_T(\mathbf{stem}(T)) \text{ and } t \in \max(W_w)\}$$

to find $\tilde{T}_t \subseteq T_w[t]$ such that $\tilde{T}_t \Vdash_{\mathbf{LT}}$ "$S \equiv^* S_{\psi_t}$" and

$$\psi = \bigcup_{w \in W} \left(\bigcup_{t \in \max(W_w)} \psi_t \restriction \tilde{T}_t \right)$$

has disjoint range. Let $\bar{T} = \bigcup_{w\in W} \bigcup_{t\in\max(W_w)} \tilde{T}_t$. Then ψ and $\bar{T}$ satisfy the lemma.

Lemma 3.4. *If $G \subseteq \mathbf{LT}$ is generic over V and in $V[G]$ and $S \in \mathcal{S}(\mathbf{LT})$ and $f : S \to 2$ then there is $T \in G$, $\psi : T \to [\omega_1]^{<\aleph_0}$ with disjoint range and $f^* : \bigcup_{t\in T} \psi(t) \to 2$, both ψ and f^* in V, such that $T \Vdash_{\mathbf{LT}}$ "$f^* \restriction \dot{S} = f$".*

Proof. The proof is the same as that of Lemma 2.9 except that in the last paragraph the following fact needs to be used: If W is a well founded tree such that $|\mathbf{succ}_W(w)| = \aleph_0$ for each $w \in W \setminus \max(W)$ whose maximal nodes are coloured in finitely many colours then there is $\bar{W} \subseteq W$ such that

- $|\mathbf{succ}_{\bar{W}}(w)| = \aleph_0$ for each $w \in \bar{W} \setminus \max(\bar{W})$
- $\mathbf{stem}(\bar{W}) = \mathbf{stem}(W)$
- $\mathbf{split}(\bar{W}) = \mathbf{split}(W) \cap (\bar{W} \setminus \max(\bar{W}))$
- all the nodes in $\max(\bar{W})$ are coloured the same colour.

It follows from the third condition that $\max(\bar{W}) \subseteq \max(W)$.

Lemma 3.5. *If $G \subseteq \mathbf{LT}$ is generic over V then $\mathcal{S}(\mathbf{LT})$ is a P-ideal.*

Proof. The fact $\mathcal{S}(\mathbf{LT})$ is closed under finite unions follows from Corollary 2.4. To see that $\mathcal{S}(\mathbf{LT})$ is closed under subsets use Lemma 3.4. If $S^* \subseteq S \in \mathcal{S}(\mathbf{LT})$ then let $f : S \to 2$ be the characteristic function of S^*. Let $T \in G$ and $\psi : T \to [\omega_1]^{<\aleph_0}$ and f^* be as in the conclusion of Lemma 3.4. Then setting $\psi^*(t) = \{\xi \in \psi(t) \mid f^*(\xi) = 1\}$ witnesses that $S^* \in \mathcal{S}(\mathbf{LT})$.

The proof of Lemma 2.8 using Lemma 3.3 instead of Lemma 2.7 can now be applied to show that $\mathcal{S}(\mathbf{LT})$ is a P-ideal. The only question the reader may have is as to why Condition (3) of Definition 3.1 is satisfied. This is easily dealt with by restricting the domain of ψ_n so that if $t \in \mathbf{domain}(\psi_n)$ then $|t| \geq n$.

Lemma 3.6. *If G is $\mathbf{LT}$ generic over V and $S \in \mathcal{S}(\mathbf{LT})$, $S \subseteq \xi \in \omega_1$, $f : S \to 2$ is a function in $V[G]$ and $Z \subseteq 2^\xi$ has full outer measure, then there is $z \in Z$ such that $f \subseteq z$.*

Proof. Given f find T, ψ and f^* satisfying the conclusion of Lemma 3.4. Let $\bar{f} = f^* \restriction \psi(\mathbf{stem}(T))$, recalling that $\psi(t) = \emptyset$ if $t \subsetneq \mathbf{stem}(T)$, and let $\mathcal{O}$ be the open set $\{h \in 2^\xi \mid h \supseteq \bar{f}\}$. Note that for a given $t \in T$ Condition (3) of the definition of $\mathcal{S}(\mathbf{LT})$ implies that there is a positive lower bound for the measure of $\{h \in 2^\xi \mid f^* \restriction \psi(s) \subseteq h\}$ as s ranges over $\mathbf{succ}_T(t)$. Therefore

$$\{h \in 2^\xi \mid |\{s \in \mathbf{succ}_T(t) \mid f^* \restriction \psi(s) \subseteq h\}| = \aleph_0\}$$

has measure one and hence, since it is independent from $\mathcal{O}$, it follows that

$$\mathcal{O} \cap \{h \in 2^\xi \mid (\forall t \in \mathbf{split}(T))\, |\{s \in \mathbf{succ}_T(t) \mid f^* \restriction \psi(s) \subseteq h\}| = \aleph_0\}$$

has positive measure and so there is some $z \in Z$ such that $\bar{f} \subseteq z$ and

$$(\forall t \in \mathbf{split}(T))\, |\{s \in \mathbf{succ}_T(t) \mid f^* \restriction \psi(s) \subseteq h\}| = \aleph_0.$$

It follows that there is $T^* \subseteq T$ such that $T^* \in \mathbf{LT}$ and such that $\mathbf{stem}(T^*) = \mathbf{stem}(T)$ and $f^* \restriction \psi(s) \subseteq z$ for all $s \in \mathbf{split}(T^*)$. Then $T^* \Vdash_{\mathbf{LT}}$ "$f \subseteq z$". It follows that there is a dense set of conditions forcing the conclusion of the lemma.

Lemma 3.7. *If W is a well founded subtree of $\omega^{<\omega}$ such that $|\mathbf{succ}_W(t)| = \aleph_0$ for every $t \in W \setminus \max(W)$ and if $\theta : \max(W) \to \omega_1$ then there is a subtree $W^* \subseteq W$ and a one-to-one function $\theta^* : \max(W^*) \to \omega_1$ such that*

- *$|\mathbf{succ}_{W^*}(t)| = \aleph_0$ for every $t \in W^* \setminus \max(W^*)$*
- *if $w \in \max(W) \cap \max(W^*)$ then $\theta^*(w) = \theta(w)$*
- *if $w \in \max(W^*) \setminus \max(W)$ then there is a subtree $W_w \subseteq W[w]$ such that $\theta(\bar{w}) = \theta^*(w)$ for each $\bar{w} \in \max(W_w)$ and such that $|\mathbf{succ}_{W_w}(t)| = \aleph_0$ for every $t \in W_w \setminus \max(W[w])$ such that $w \subseteq t$.*

Proof. This is standard argument by induction on the rank of the tree W.

Lemma 3.8. *If G is $\mathbf{LT}$ generic over V then no uncountable subset of ω_1 is orthogonal to $\mathcal{S}(\mathbf{LT})$ in $V[G]$.*

Proof. This is similar to the proof of Lemma 2.11. Given $T \in \mathbf{LT}$ such that $T \Vdash_{\mathbf{LT}}$ "$Z \in [\omega_1]^{\aleph_1}$" proceed by induction on n to find $T_{n+1} \leq_n T_n$ (with $T_0 = T$) such that for each $i \leq n$ there is a well founded tree $W_i \subseteq T_{n+1}\langle u_i\rangle$ such that $\max(W_i)$ is a front in $T_{n+1}\langle u_i\rangle$ and a one-to-one function $\psi_i : \max(W_i) \to \omega_1$ such that $T_{n+1}[w] \Vdash_{\mathbf{LT}}$ "$\psi_i(w) \in Z$" for each $w \in \max(W_i)$. To carry out the induction, given T_n, let σ be a name for the least element of Z greater than all elements of

$$\bigcup_{i \in n} \left(\bigcup_{w \in \max(W_i)} \psi_i(w) \right)$$

and then find $T^* \subseteq T_n\langle u_n\rangle$ such that $\mathbf{stem}(T^*) = \Psi_{T_n}(u_n)$ and there is a well founded tree $W^* \subseteq T^*$ such that $\max(W^*)$ is a front in T^* and $\theta : \max(W^*) \to \omega_1$ such that $T^*[w] \Vdash_{\mathbf{LT}}$ "$\sigma = \theta(w)$" for each $w \in \max(W^*)$. Then apply Lemma 3.7 to yield $W_{n+1} \subseteq W^*$ and θ^* and let $T_{n+1} = \bigcup_{w \in \max(W_{n+1})} T^*[w]$ and then let $\psi_{n+1} = \theta^*$. Then let $\bar{T} = \bigcap_{n \in \omega} T_n$ and define ψ by $\psi(t) = \{\psi_j(t) \mid t \in \max(W_j)\}$ and note that

$$|\psi(t)| \leq |\{j \in \omega \mid t \in \max(W_j)\}| \leq |t|$$

as required by Condition (3) of the definition of of $\mathcal{S}(\mathbf{LT})$. It follows that $\bar{T} \Vdash_{\mathbf{LT}}$ "$S_\psi \in [Z]^{\aleph_0}$".

Theorem 3.9. *Let V be a model of set theory and suppose that $U : \omega_1^2 \to 2$ is a symmetric, measure saturated function in V and that $G \subseteq \mathbf{LT}$ is generic over V. In $V[G]$ let $H \subseteq \mathbb{P}_{\mathcal{S}(\mathbf{LT})}$ be generic over $V[G]$. Then in $V[G][H]$ the function U is universal.*

Proof. This is the same as the proof of Theorem 2.13 using measure in the place of category.

Corollary 3.10. *It is consistent with $\mathfrak{b} = \mathfrak{d} = \aleph_2$ that there is a universal graph on ω_1.*

Proof. This the same as the proof of Corollary 2.14.

4. Measure saturated graphs are universal after adding ω^ω-bounding reals

This section contains the key consistency result needed to establish the main result of this article, namely that the existence of a universal graph on ω_1 does not entail the existence of a universal function from ω_1^2 to ω. The key concepts are already contained in §3 the only difference being that the partial order $\mathbf{PT}_{f,g}$ of Definition 7.3.3 of [8] will be used in the place of $\mathbf{LT}$. As can be seen in the exposition of $\mathbf{PT}_{f,g}$ in [8], there are a great many analogies between this partial order and $\mathbf{LT}$ and these explain the similarities between §3 and this section. The following definition introduces the key technical notion.

Definition 4.1. The partial order $\mathbf{PT}_{f,g}$ will be used with the functions f and g defined as follows. First let $a_n > 0$ be such that $\sum_{n=0}^{\infty} a_n < 1$. Let $g(0,0) = 1$. If $g(n,n)$ has been defined let $f(n) = \max(g(n,n), 2^n)$. Then let $g(n+1,0) = 1$ and then choose $g(n+1,k+1)$ be so large that if

- $[X_{i,j}]_{i\in g(n+1,k+1), j\in n+1}$ is a matrix of independent 2-valued random variables
- the probability that $X_{i,j} = 1$ is 1/2
- $\varphi : g(n+1,k+1) \times (n+1) \to 2$

then the probability that

$$|\{i \in g(n+1,k+1) \mid (\forall j \in n+1)\ X_{i,j} = \varphi(j)\}| \geq g(n+1,k) \tag{4.1}$$

is greater than $1 - a_n / \prod_{m=0}^{n} f(m)$. It will also be required that the following inequalities hold:

$$g(n,j+1) > g(n,j)^2 \tag{4.2}$$

$$g(n+1,j+1) > g(n+1,j)f(n) \tag{4.3}$$

$$g(n,j+1) > g(n,j)\left(\prod_{m\in n+1} f(m)\right)^2 \tag{4.4}$$

$$g(n,j+1) > g(n,j) + \sum_{j\leq n} j \left(\prod_{m\in j+1} f(m)\right)^2 \tag{4.5}$$

$$g(n,j+1) > g(n,j) \prod_{m\in n+1} 2^m \tag{4.6}$$

all of which are easily obtained. The functions f and g are fixed throughout this section.

Recall that $\mathbf{PT}_{f,g}$ consists of trees $T \subseteq \bigcup_{n\in\omega} \prod_{i\in n} f(i)$ such that there is a function $r : \omega \to \omega$ satisfying that $\lim_{n\to\infty} r(n) = \infty$ and such that

$$|\mathbf{succ}_T(t)| > g(|t|, r(|t|))$$

for all $t \in T$. For any $T \in \mathbf{PT}_{f,g}$ fix $r_T : \omega \to \omega$ witnessing that $T \in \mathbf{PT}_{f,g}$. The ordering on $\mathbf{PT}_{f,g}$ is inclusion.

Definition 4.2. If $G \subseteq \mathbf{PT}_{f,g}$ is generic over V define $\mathcal{S}(\mathbf{PT}_{f,g})$ to be the set of all $S \subseteq \omega_1$ such that there is $T \in G$ and $\psi : T \to [\omega_1]^{<\aleph_0}$ such that

1. $\psi \in V$
2. if $s \neq t$ then $\psi(s) \cap \psi(t) = \emptyset$
3. $\lim_{t\in T} |\psi(t)|/|t| = 0$
4. $T \Vdash_{\mathbf{PT}_{f,g}}$ "$\dot{S} = S_\psi$".

recalling that S_ψ is defined in Definition 2.1. As in Definition 3.1 it will be assumed that $\psi(s) = \emptyset$ if $s \subsetneq \mathbf{stem}(T)$.

Notation 4.3. In this section it will be convenient, for $k \in \omega$ and any tree T, to use the notation $T(k) = \{t \in T \mid |t| = k\}$ and to note that $|T(k)| \leq \prod_{j\in k} f(j)$ if $T \in \mathbf{PT}_{f,g}$.

The following is the lemma that corresponds to Lemma 2.6 in the context of the partial order $\mathbf{PT}_{f,g}$.

Lemma 4.4. *If* $\{T_i\}_{i\in m}$ *and* $\{\psi_i\}_{i\in m}$ *satisfy the following*

- $m \le \prod_{j\in n+1} f(j)$
- $T_i \in \mathbf{PT}_{f,g}$
- $|\mathbf{stem}(T_i)| = n$ *for each* i
- $r_{T_i}(\ell) > 2$ *for each* $i \in m$ *and* $\ell \ge n$
- *each* $\psi_i : T_i \to [\omega_1]^{<\aleph_0}$ *satisfies the requirements of Definition 4.2*

then there are $T_i^* \subseteq T_i$ *such that*

$$(\forall t \in T_i^*)\ |\mathbf{succ}_{T_i^*}(t)| \ge g(|t|, r_{T_i}(|t|) - 2) \tag{4.7}$$

$$(\forall i < j < m)(\forall t \in T_i^*)(\forall s \in T_j^*)\ \text{ if } |t| > n \text{ and } |s| > n \text{ then } \psi_i(t) \cap \psi_j(s) = \emptyset. \tag{4.8}$$

Proof. Begin by noting the following simple fact: If $\mathcal{F}_i \in [\omega_1]^{<k}$ is a pairwise disjoint family for each $i \in u$ then there are $\mathcal{G}_i \subseteq \mathcal{F}_i$ such that $\bigcup_{i\in u} \mathcal{G}_i$ is a pairwise disjoint family and $|\mathcal{G}_i| \ge |\mathcal{F}_i|/uk$. Given $k \ge n$, define $\mathcal{F}^k_{t,i,} = \{\psi_i(s) \mid s \in \mathbf{succ}_{T_i}(t)\}$ for each $i \in m$ and $t \in T_i(k)$. Using Inequality (4.4), for each $k \ge n$ choose $\mathcal{G}^k_{t,i} \subseteq \mathcal{F}^k_{t,i}$ such that $|\mathcal{G}^k_{t,i}| \ge g(k, r_{T_i}(k) - 1)$ and

$$\mathcal{Z}_k = \bigcup_{i\in m}\left(\bigcup_{t\in T_i(k)} \mathcal{G}^k_{t,i}\right)$$

is a pairwise disjoint family for each $k \ge n$. Using the fact that $|\bigcup \mathcal{Z}_k| < k(\prod_{j\in k} f(j))^2$ and Inequality (4.5) it is possible to find $\mathcal{H}_{t,i} \subseteq \mathcal{G}^{|t|}_{t,i}$ such that

$$\bigcup_{k\in\omega}\bigcup_{i\in m}\left(\bigcup_{t\in T_i(k)} \mathcal{H}_{t,i}\right)$$

is a pairwise disjoint family and such that $|\mathcal{H}_{t,i}| \ge g(|t|, r_{T_i}(|t|) - 2)$ if $t \in T_i$. It is then immediate that there are $T_i^* \subseteq T_i$ such that $\mathbf{stem}(T_i^*) = \mathbf{stem}(T_i)$ and such that $\mathbf{succ}_{T_i^*}(t) = \mathcal{H}_{t,i}$ if $t \supseteq \mathbf{stem}(T_i^*)$ and $t \in T_i^*$. These T_i^* satisfy the lemma.

Corollary 4.5. $\mathcal{S}(\mathbf{PT}_{f,g})$ *is closed under finite unions.*

Proof. This follows from an argument similar to that of Lemma 4.4 using the following fact: If

$$\{a_{i,j}\}_{i\in u, j\in n} \subseteq [\omega_1]^{<k}$$

is such that for each $i \in u$ the family $\{a_{i,j}\}_{j\in n}$ is pairwise disjoint then there is $Y \subseteq n$ such that $|Y| \ge n/uk$ and the family $\{\bigcup_{i\in u} a_{i,j}\}_{j\in Y}$ is pairwise disjoint. Of course, it is also necessary to note that the limit in Condition (3) of Definition 4.2 is preserved by finite sums.

Lemma 4.6. *If* $T \in \mathbf{PT}_{f,g}$ *is such that*

$$T \Vdash_{\mathbf{PT}_{f,g}} \text{“}S \in \mathcal{S}(\mathbf{PT}_{f,g})\text{”} \tag{4.9}$$

$$M \in \omega \text{ is such that } (\forall \ell \ge |\mathbf{stem}(T)|)\ r_T(\ell) > M \tag{4.10}$$

then there is $T^* \subseteq T$ *and* $\psi : T^* \to [\omega_1]^{<\aleph_0}$ *such that*

- ψ *has disjoint range and satisfies Condition (3) of Definition 4.2*
- $\mathbf{stem}(T) = \mathbf{stem}(T^*)$
- $T^* \Vdash_{\mathbf{PT}_{f,g}}$ “$S \equiv^* S_\psi$”
- $|\mathbf{succ}_{T^*}(t)| \ge g(|t|, M)$ *for all* $t \in T^*$ *such that* $t \supseteq \mathbf{stem}(T)$.

PROOF. Begin by observing that for all $T \in \mathbf{PT}_{f,g}$ there exists $T' \subseteq T$ such that

$$(\exists \psi : T \to [\omega_1]^{<\aleph_0})\ \psi \text{ satisfies Condition (3) of Definition 4.2 and}$$
$$T' \Vdash_{\mathbf{PT}_{f,g}} \text{“} S \equiv^* S_\psi\text{” and } \psi(s) = \emptyset \text{ if } s \subseteq \mathbf{stem}(T)$$
$$\text{and } (\forall s \in T')\ \text{ if } s \supseteq \mathbf{stem}(T') \text{ then } |\mathbf{succ}_{T'}(s)| \geq g(|s|, M+2). \quad (4.11)$$

To see this simply find $\bar{T} \subseteq T$ and $\tilde{T}$ and $\bar{\psi} : \tilde{T} \to [\omega_1]^{<\aleph_0}$ such that $\bar{T} \Vdash_{\mathbf{PT}_{f,g}}$ “$\tilde{T} \in G$ and $S \equiv^* S_{\bar{\psi}}$”. Note that $\bar{T} \cap \tilde{T} \in \mathbf{PT}_{f,g}$ because $\bar{T} \Vdash_{\mathbf{PT}_{f,g}}$ “$\tilde{T} \in G$”. Choose ℓ such that $r_{\bar{T}\cap\tilde{T}}(i) > M+2$ for all $i \geq \ell$ and let $t \in \bar{T} \cap \tilde{T}(\ell)$. Let $T' = (\bar{T} \cap \tilde{T})[t]$ and define ψ with domain T' by

$$\psi(s) = \begin{cases} \emptyset & \text{if } s \subseteq t \\ \bar{\psi}(s) & \text{otherwise.} \end{cases}$$

Now define $\rho(t) = 0$ if there is $T' \subseteq T[t]$ such that $T \in \mathbf{PT}_{f,g}$, $\mathbf{stem}(T) = t$ and (4.11) holds. Define

$$\rho(t) = \min\left(\{\alpha + 1 \mid |\{s \in \mathbf{succ}_T(t) \mid \rho(s) \leq \alpha\}| \geq g(|t|, r(|t|) - 1)\}\right)$$

and note that (4.11) and a standard argument show that $\rho(\mathbf{stem}(T))$ is defined. There is no harm in assuming that $r_T(|\mathbf{stem}(T)|) > 1$.

It follows that there is some $\hat{T} \subseteq T$ and $k \in \omega$ such that

$$(\forall s \in \hat{T}) \text{ if } |s| < k \text{ then } |\mathbf{succ}_{\hat{T}}(s)| \geq g(|s|, r_T(|s|) - 1)$$

and, moreover, for each $t \in \hat{T}(k)$ there is $\psi_t : \hat{T}[t] \to [\omega_1]^{<\aleph_0}$ with disjoint range such that ψ_t satisfies Condition (3) of Definition 4.2 and

$$\hat{T}[t] \Vdash_{\mathbf{PT}_{f,g}} \text{“} S \equiv^* S_{\psi_t}\text{”} \quad (4.12)$$
$$(\forall j \leq k)\ \psi_t(t \restriction j) = \emptyset \quad (4.13)$$
$$(\forall s \in \hat{T}[t])\ \text{ if } s \supseteq t \text{ then } |\mathbf{succ}_{\hat{T}}(s)| \geq g(|s|, M+2). \quad (4.14)$$

Now use Lemma 4.4 to find $T^t \subseteq \hat{T}[t]$ such that

$$\bigcup_{t \in T(k)} \psi_t \restriction T^t$$

is a function with disjoint range and $|\mathbf{succ}_{T^t}(s)| \geq g(|s|, M)$ for all $s \in T^t$ such that $s \supseteq t$. Let $T^* = \bigcup_{t \in T(k)} \psi_t \restriction T^t$.

Corollary 4.7. *If $T \in \mathbf{PT}_{f,g}$ and*

$$T \Vdash_{\mathbf{PT}_{f,g}} \text{“}(\forall n \in \omega)\ S_n \in \mathcal{S}(\mathbf{PT}_{f,g})\text{”}$$

then there is $T^* \subseteq T$ and $\psi : T^* \to [\omega_1]^{<\aleph_0}$ with disjoint range satisfying Condition (3) of Definition 4.2 such that

$$T^* \Vdash_{\mathbf{PT}_{f,g}} \text{“}(\forall n \in \omega)\ S_n \subseteq^* S_\psi\text{”}$$

and, hence, $1 \Vdash_{\mathbf{PT}_{f,g}}$ “$\mathcal{S}(\mathbf{PT}_{f,g})$ is a P-ideal”.

PROOF. The fact that $\mathcal{S}(\mathbf{PT}_{f,g})$ is closed under finite unions was established in Corollary 4.5. Use Lemma 4.11 to prove that it is closed under subsets in the same way that Lemma 3.4 was used in Lemma 3.5 to prove the corresponding fact about $\mathcal{S}(\mathbf{LT})$.

Inductively find T_n, K_n and ψ_n such that

- $T_{n+1} \subseteq T_n$
- $T_{n+1}(K_n) = T_n(K_n)$
- $\psi_n : \{t \in T_n \mid |t| > K_n\} \to [\omega_1]^{<\aleph_0}$ has disjoint range
- $|\psi_n(t)|/|t| < 1/n$ if $t \in T_n$ and $|t| > K_n$

- $\psi_{n+1}(t) \supseteq \psi_n(t)$ for all t in the domain of ψ_{n+1}
- $|\mathbf{succ}_{T_n}(t)| \geq g(|t|, n)$ for $t \in T_n$ if $|t| > K_n$
- $K_{n+1} > K_n \geq n$
- $T_n \Vdash_{\mathbf{PT}_{f,g}}$ "$S_n \equiv^* S_{\psi_n}$".

and then let $T^* = \bigcap_{n\in\omega} T_n$ and define $\psi(t) = \bigcup_{n\in\omega} \psi_n(t)$. Observe that t is in the domain of only finitely many ψ_n and, moreover, if k is maximal such that $t \in \mathbf{domain}(\psi_k)$ then $|\psi_k(t)|/|t| < 1/k$ and $\psi_k(t) \supseteq \psi_j(t)$ for any j such that $t \in \mathbf{domain}(\psi_j)$. Hence $\lim_{t\in T^*} |\psi(t)|/|t| = 0$.

At stage n choose K_{n+1} so large that $r_{T_n}(j) \geq n+4$ for all $j \geq K_{n+1}$. Then for all $t \in T_n(K_{n+1})$ use Lemma 4.6 to find $T^*_t \subseteq T_n[t]$ such that $|\mathbf{succ}_{T^*_t}(s)| \geq g(|s|, n+3)$ for all $s \in T^*_t$ such that $s \supseteq t$ and $\psi_t : T^*_t \to [\omega_1]^{<\aleph_0}$ with disjoint range such that

$$T^*_t \Vdash_{\mathbf{PT}_{f,g}} \text{“}S_{n+1} \equiv^* S_{\psi_t}\text{”}$$

and such that $\psi_t(s) = \emptyset$ if $|s| \leq K_{n+1}$. Then use Lemma 4.4 to find $T^{**}_t \subseteq T^*_t$ satisfying Conditions 4.7 and 4.8. In particular, $r_{T^{**}_t}(j) \geq n+1$ if $j \geq |t|$. Then use the argument of Corollary 4.5 to find $T'_t \subseteq T^{**}_t$ such that $r_{T'_t}(j) \geq n$ if $j \geq |t|$ and such that the function ψ_{n+1} defined by $\psi_{n+1}(s) = \psi_t(s) \cup \psi_n(s)$ if $s \in T'_t$ and $t \in T_n(K_{n+1})$ has disjoint range. Let

$$T_{n+1} = \bigcup_{t\in T_n(K_{n+1})} T'_t$$

and and observe that all of the induction hypotheses are satisfied.

The following lemma is the counterpart to Lemma 3.7 in the context of $\mathbf{PT}_{f,g}$.

Lemma 4.8. *If W is a finite subtree of $\bigcup_{n\in\omega} \prod_{i\in n} f(i)$ such that $|\mathbf{succ}_W(t)| \geq g(|t|, r(|t|))$ for every $t \in W \setminus \max(W)$ and if $\theta : \max(W) \to \omega_1$ then there is a subtree $W^* \subseteq W$ and a one-to-one function $\theta^* : \max(W^*) \to \omega_1$ such that*

- *$|\mathbf{succ}_{W^*}(t)| \geq g(|t|, r(|t|) - 2)$ for every $t \in W^* \setminus \max(W^*)$*
- *if $w \in \max(W) \cap \max(W^*)$ then $\theta^*(w) = \theta(w)$*
- *if $w \in \max(W^*) \setminus \max(W)$ then there is a subtree $W_w \subseteq W[w]$ such that $\theta(\bar{w}) = \theta^*(w)$ for each $\bar{w} \in \max(W_w)$ and such that $|\mathbf{succ}_{W_w}(t)| \geq g(|t|, r(|t|) - 2)$ for every $t \in W_w \setminus \max(W[w])$ such that $w \subseteq t$.*

Proof. Proceed by induction on the difference between the height of W and the cardinality of $\mathbf{stem}(W)$, the case when $\mathbf{stem}(W)$ is the maximal element of W being trivial. Given the result for n let W be a tree such that the difference between the height of W and the cardinality of $s_W = \mathbf{stem}(W)$ is $n+1$. For each $s \in \mathbf{succ}_W(s_W)$ the difference between the the height of $W[s]$ and the cardinality of $s = \mathbf{stem}(W[s])$ is n and hence there are W^*_s and θ^*_s witnessing that the lemma holds.

Let $A = \{s \in \mathbf{succ}_W(s_W) \mid \max(W^*_s) = \{s\}\}$. The first case to consider is that

$$|A| \geq g(|s_W|, r(s_W) - 1).$$

Suppose first that there is $\alpha \in \omega_1$ such that if $B_\alpha = \{s \in \mathbf{succ}_W(s_W) \mid \theta^*_s(s) = \alpha\}$ then

$$|B_\alpha| \geq g(|s_W|, r(s_W) - 2).$$

Then let W^* be the tree with maximal element s_W and let $\theta^*(s_W) = \alpha$. Let $W_{s_W} = \bigcup_{s\in B_\alpha} W_s$.

On the other hand, if $|B_\alpha| < g(|s_W|, r(s_W) - 1)$ for each $\alpha \in \omega_1$ then use Inequality (4.2) to conclude that there is $C \subseteq A$ such that $|C| \geq g(|s_W|, r(s_W) - 2)$ and the mapping $s \mapsto \theta^*_s(s)$ is one-to-one on C. In this case let W^* be the tree whose set of maximal elements is precisely C. For each $s \in C$ the set W_s is already defined from the induction hypothesis.

Now suppose that $|A| < g(|s_W|, r(s_W) - 1)$. Observe that if $s \in \mathbf{succ}_W(s_W) \setminus A$ and $w \in W^*_s$ is such that $\mathbf{succ}_{W^*_s}(w) \subseteq \max(W^*_s)$ then $|w| > |s_W|$ and hence by Inequality (4.3) it follows that

$$g(|w|, r(|w|)) > g(|w|, r(|w|) - 1) f(|s_W|) \geq g(|w|, r(|w|) - 1) |\mathbf{succ}_W(s_W)|.$$

From this it is easy to find for each $w \in W^*_s$ such that $\mathbf{succ}_{W^*_s}(w) \subseteq \max(W^*_s)$ a set $Z_w \subseteq \mathbf{succ}_{W^*_s}(w)$ such that $|Z_w| > g(|w|, r(|w|) - 1)$ and such that

$$\theta^* = \bigcup_{s \in \mathbf{succ}_W(s_W) \setminus A} \bigcup \left\{ \theta^*_s \restriction Z_w \;\middle|\; w \in W^*_s \text{ and } \mathbf{succ}_{W^*_s}(w) \subseteq \max(W^*_s) \right\}$$

is a one-to-one function. Let W^* be the tree whose set of maximal elements is

$$\bigcup_{s \in \mathbf{succ}_W(s_W) \setminus A} \bigcup \left\{ Z_w \;\middle|\; w \in W^*_s \text{ and } \mathbf{succ}_{W^*_s}(w) \subseteq \max(W^*_s) \right\}.$$

The trees W_t for $t \in Z_w \subseteq \max(W^*)$ are already defined from the induction hypothesis.

Lemma 4.9. *If G is $\mathbf{PT}_{f,g}$ generic over V then every infinite subset of ω_1 is orthogonal to $\mathcal{S}(\mathbf{PT}_{f,g})$ in $V[G]$.*

Proof. This is similar to the proof of Lemma 3.8 but using Lemma 4.8.

The stronger conclusion of Lemma 4.9 that all infinite sets are orthogonal to $\mathcal{S}(\mathbf{PT}_{f,g})$, as opposed to uncountable sets in the conclusion of Lemma 3.8, will not be used here, but seems worthy of note in any case.

Lemma 4.10. *Let $T \in \mathbf{PT}_{f,g}$, $K = |\mathbf{stem}(T)|$ and $k = \prod_{m \in K} 2^m$. Then for any $J \geq K$ and $\theta : T(J) \to k$ there is $W \subseteq T$ such that $\max(W) \subseteq T(J)$ and $\theta \restriction max(W)$ is constant and such that $|\mathbf{succ}_W(t)| \geq g(|t|, r_T(|t|) - 1)$ for every $t \in W \setminus max(W)$.*

Proof. Proceed by induction on $J - K$ noting that if $J = K$ the result is trivial. If the result it true for n let T and J be given such that $J - K = n + 1$. Since $k < \prod_{m \in K+1} 2^m$ it follows from the induction hypothesis that there is $W_s \subseteq T[s]$ such that $\max(W_s) \subseteq T[s](J)$ and such that $\theta \restriction \max(W_s)$ is constant with value v_s and such that $|\mathbf{succ}_{W_s}(t)| \geq g(|t|, r(|t|) - 1)$ for each $s \in \mathbf{succ}_T(\mathbf{stem}(T))$ and $t \in W_s \setminus \max(W_s)$. Now use Inequality (4.6) of Definition 4.1 to find $Z \subseteq \mathbf{succ}_T(\mathbf{stem}(T))$ and v such that $v_s = v$ for $s \in Z$ and such that $|Z| \geq g(K, r(K) - 1)$. Then $W = \bigcup_{z \in Z} W_z$ is the required tree.

Lemma 4.11. *If $T \in \mathbf{PT}_{f,g}$ and*

$$T \Vdash_{\mathbf{PT}_{f,g}} \text{“}S \in \mathcal{S}(\mathbf{PT}_{f,g}) \text{ and } f : S \to 2\text{”}$$

then there are $T^* \subseteq T$ and $\psi : T^* \to [\omega_1]^{<\aleph_0}$ such that $T^* \Vdash_{\mathbf{PT}_{f,g}}$ “$S \equiv^* S_\psi$” and there is

$$f^* : \bigcup_{t \in T^*} \psi(t) \to 2$$

such that $T^* \Vdash_{\mathbf{PT}_{f,g}}$ “$f \equiv^* f^* \restriction S$”.

Proof. Begin by using Lemma 4.6 to find $\bar{T} \subseteq T$ and $\psi : \bar{T} \to [\omega_1]^{<\aleph_0}$ such that $\bar{T} \Vdash_{\mathbf{PT}_{f,g}}$ “$S \equiv^* S_\psi$”. A standard rank argument can then be used to find $T^{**} \subseteq \bar{T}$ and an increasing sequence of positive integers $\{L_i\}_{i \in \omega}$ such that $T^{**} \in \mathbf{PT}_{f,g}$ and for each $t \in T^{**}(L_{i+1})$ there is $f_t : \bigcup_{m \in L_i} \psi(t \restriction m) \to 2$ such that

$$T^{**}[t] \Vdash_{\mathbf{PT}_{f,g}} \text{“}f_t = f \restriction \left(\bigcup_{m \in L_i} \psi(t \restriction m) \right)\text{”}.$$

Then use Lemma 4.10 to find T^* such that $r_{T^*}(n) \geq r_{T^{**}}(n) - 1$ and such that $f_t = f_s$ if $|s| = |t| = L_{i+1}$ and $s \restriction L_i = t \restriction L_i$.

Lemma 4.12. *If G is $\mathbf{PT}_{f,g}$ generic over V and $S \in \mathcal{S}(\mathbf{PT}_{f,g})$, $S \subseteq \xi \in \omega_1$, $f : S \to 2$ is a function in $V[G]$ and $Z \subseteq \omega^\xi$ has full outer measure, then there is $z \in Z$ such that $f \subseteq z$.*

Proof. Given f find T, ψ and f^* satisfying the conclusion of Lemma 4.11. Let $\bar{f}$ be a finite partial function such that $\bar{f} \supseteq \psi(\mathbf{stem}(T))$ and

$$(f^* \restriction S \setminus \mathbf{domain}(\bar{f})) \cup \bar{f} = f$$

and let $\mathcal{O}$ be the open set $\{h \in 2^\xi \mid h \supseteq \bar{f}\}$. Then $Z \cap \mathcal{O}$ has positive outer measure. Note that Condition 4.1 of Definition 4.1 implies that for any $t \in T$ the probability that

$$\{h \in 2^\xi \mid |\{s \in \mathbf{succ}_T(t) \mid f^* \restriction \psi(s) \subseteq h\}| \geq g(|t|, r_T(|t|) - 1)\}$$

is greater than $1 - a_{|t|}/\prod_{m=0}^{|t|} f(m)$ and hence, for any n the probability that

$$\{h \in 2^\xi \mid (\forall t \in T(n))\, |\{s \in \mathbf{succ}_T(t) \mid f^* \restriction \psi(s) \subseteq h\}| \geq g(|t|, r_T(|t|) - 1)\}$$

is greater than $1 - a_{|t|}$. It follows from the choice of the a_n that the probability that

$$\{h \in 2^\xi \mid (\forall t \in T)\, |\{s \in \mathbf{succ}_T(t) \mid f^* \restriction \psi(s) \subseteq h\}| \geq g(|t|, r_T(|t|) - 1)\}$$

is positive and, morever, this set is independent from $\mathcal{O}$. It follows that there is some $z \in Z$ such that

$$(\forall t \in T)\, |\{s \in \mathbf{succ}_T(t) \mid f^* \restriction \psi(s) \subseteq z\}| \geq g(|t|, r(|t|) - 1)$$

and $\bar{f} \subseteq z$. It follows that there is $T^* \subseteq T$ such that $T^* \in \mathbf{PT}_{f,g}$ and such that if $t \in T$ then $f^* \restriction \psi(t) \subseteq z$. Then $T^* \Vdash_{\mathbf{PT}_{f,g}}$ "$\dot{f} \subseteq z$".

Theorem 4.13. *Let V be a model of set theory and suppose that $U : \omega_1^2 \to 2$ is a symmetric, measure saturated function in V and that $G \subseteq \mathbf{PT}_{f,g}$ is generic over V. In $V[G]$ let $H \subseteq \mathbb{P}_{\mathcal{S}(\mathbf{PT}_{f,g})}$ be generic over $V[G]$. Then in $V[G][H]$ the graph U is universal.*

Proof. This is the same as the proof of Theorem 3.9.

Corollary 4.14. *It is consistent that $\mathfrak{b} = \mathfrak{d} = \aleph_1$ and there is a universal graph on ω_1.*

Proof. This the same as the proof of Corollary 2.14 but using that $\mathbf{PT}_{f,g}$ is ω^ω bounding.

5. Tree Partial Orders with Additional Structure

5.1. The general framework

The results of the preceding sections were originally obtained by a more complicated argument that was eventually replaced by the simpler arguments described in §2, §3 and §4. However, there is one result for which the simplified argument does not seem to be sufficient; this is the question of finding a universal function from ω_1^2 to ω rather than a universal function from ω_1^2 to 2. This section will describe an argument showing that it is consistent with $\mathfrak{b} < \mathfrak{d}$ that there is a universal function from ω_1^2 to ω. The argument has wider applicability though, which motivated readers will be able to find on their own.

Definition 5.1. Let $G_0 : \omega_1^2 \to \omega$ and $G_1 : \omega_1^2 \to \omega$ be symmetric functions and let $\mathcal{E}(G_0, G_1)$ denote the set of all finite, one-to-one functions e that are isomorphisms between $G_1 \restriction \mathbf{domain}(e)^2$ and $G_0 \restriction \mathbf{range}(e)^2$; in other words, $G_1(\eta, \zeta) = G_0(e(\eta), e(\zeta))$ all distinct η and ζ in the domain of e.

Definition 5.2. If $G_0 : \omega_1^2 \to \omega$ and $G_1 : \omega_1^2 \to \omega$ are symmetric functions and $T \subseteq \omega^{<\omega}$ is a tree then a function $E : T \to \mathcal{E}(G_0, G_1)$ will be called *good* if:

(a) if s and t belong to T and $s \subseteq t$ then $E(s) \subseteq E(t)$

(b) if s and t belong to T then $\mathbf{range}(E(t)) \cap \mathbf{range}(E(s)) = \mathbf{range}(E(s \cap t))$.

The following definitions will be used only in the context of $\mathbb{P} = \mathbf{PT}$ but it seems worth providing the more general context since the definitions are applicable for any partial order consisting of trees ordered by inclusion.

Definition 5.3. Let $\mathbb{P}$ be a tree partial order. If $G_0 : \omega_1^2 \to \omega$ and $G_1 : \omega_1^2 \to \omega$ are symmetric functions define $\mathbb{P}_{G_0,G_1}$ to consist of triples (T, E, η) such that

1. $T \in \mathbb{P}$

2. $E : T \to \mathcal{E}(G_0, G_1)$ is good

3. $\eta \in \omega_1$.

If $p = (T, E, \eta) \in \mathbb{P}_{G_0,G_1}$ the notation (T^p, E^p, η^p) will be used to denote (T, E, η). Define $p \leq q$ if and only if

4. $T^p \subseteq T^q$

5. $E^p(t) = E^q(t)$ for each $t \in T^p$ such that $t \subsetneq \mathbf{stem}(T^p)$

6. $E^p(t) \supseteq E^q(t)$ for each $t \in T^p$ such that $t \supseteq \mathbf{stem}(T^p)$

7. $(\mathbf{range}(E^p(t)) \setminus \mathbf{range}(E^q(t))) \cap \eta^q = \emptyset$ for all $t \in T^p$

8. $\eta^p \geq \eta^q$.

Definition 5.4. If $G \subseteq \mathbb{P}_{G_0,G_1}$ is generic define $E_G : \omega_1 \to \omega_1$ by $E_G = \bigcup_{p \in G} E(\mathbf{stem}(T^p))$.

It is immediate that E_G is a partial embedding of G_1 into G_0. However, some extra requirements will be needed to guarantee that E_G is a total embedding. The following arguments restrict attention to Miller forcing.

5.2. Modifying Miller forcing

The first thing to check is that $\mathbf{PT}_{G_0,G_1}$ is proper, indeed, that it satisfies Axiom A. This will rely on the partial orders $\prec_n$ defined in Definition 7.3.44 of [8]. However, the proof is not immediate as this is the place at which η^p, the third component of $p \in \mathbf{PT}_{G_0,G_1}$, is used.

Lemma 5.5. *If $G_0 : \omega_1^2 \to \omega$ and $G_1 : \omega_1^2 \to \omega$ are symmetric functions then $\mathbf{PT}_{G_0,G_1}$ satisfies Axiom A.*

PROOF. For p and q in $\mathbf{PT}_{G_0,G_1}$ define $p \prec_n q$ if

1. $p \leq q$

2. $\mathbf{split}_n(T^p) = \mathbf{split}_n(T^q)$

3. $E^p(t \restriction k) = E^q(t \restriction k)$ for all $t \in \mathbf{split}_n(T^p)$ and $k \leq |t|$.

It needs to be verified that these partial orders witness that $\mathbf{PT}_{G_0,G_1}$ satisfies Axiom A. The only point that needs an argument is that given a dense $D \subseteq \mathbf{PT}_{G_0,G_1}$, $p \in \mathbf{PT}_{G_0,G_1}$ and $n \in \omega$ there is $q \prec_n p$ such that for each $t \in \mathbf{split}_n(T^p) = \mathbf{split}_n(T^q)$ the condition $(T^q[t], E^q \restriction T^q[t], \eta^q) \in D$.

To see that this is so, let $\{t_k\}_{k \in \omega}$ enumerate $\mathbf{split}_n(T^p)$. Construct inductively $p_k \in \mathbf{PT}_{G_0,G_1}$ and $\eta_k \in \omega_1$ such that:

- $\eta_0 = \eta^p$
- $p_k \leq (T^p[t_k], E^p \restriction T^p[t_k], \eta_k)$
- $p_k \in D$
- $\eta_k \supseteq \bigcup_{i \in k} \bigcup_{s \in T^{p_i}} \mathbf{range}(E^{p_i}(s))$
- $\eta_k \geq \eta^{p_i}$ for $i \in k$.

Then let $T^q = \bigcup_{k \in \omega} T^{p_k}$ and $E^q = \bigcup_{k \in \omega} E^{p_k}$ and $\eta^q = \bigcup_{k \in \omega} \eta^{p_k}$ and set $q = (T^q, E^q, \eta^q)$. Note that Condition (3) holds because of (5) in Definition 5.3.

Lemma 5.6. *If $G_0 : \omega_1^2 \to \omega$ is category saturated and $\xi \in \omega_1$ then*

$$\{p \in \mathbf{PT}_{G_0,G_1} \mid \xi \in \mathbf{domain}(E^p(\mathbf{stem}(T^p)))\}$$

is dense in $\mathbf{PT}_{G_0,G_1}$.

Proof. Now let $p \in \mathbf{PT}_{G_0,G_1}$. For $s \in \mathbf{split}_m(T^p)$ let

$$\mathcal{F}(s) = \left\{ E^p(t) \setminus E^p(s) \;\middle|\; t \in \mathbf{split}_{m+1}(T) \text{ and } t \supseteq s \right\}$$

and note that any two distinct functions in $\mathcal{F}(s)$ have disjoint ranges by (b) of Definition 5.2. For $f \in \mathcal{F}(s)$ define $F_{\xi,f} : \mathbf{range}(f) \to 2$ by $F_{\xi,f}(f(\zeta)) = G_1(\zeta,\xi)$.

Let $\mu \in \omega_1$ be so large that the range of each $E^p(t)$ is contained in μ. Note that if

$$\mathcal{D}(s) = \{h \in 2^\mu \mid h \supseteq F_{\xi,f} \text{ for infinitely many } f \in \mathcal{F}(s)\}$$

then $\mathcal{D}(s)$ is dense a G_δ subset of 2^μ with the product topology. Since G_0 is category saturated it is possible to find $\xi^* > \eta^p$ such that $G_0^{\xi^*} \restriction \mu \in \mathcal{D}(s)$ for all $s \in \mathbf{split}(T^p)$ where $G_0^{\xi^*}$ is defined in Definition 1.3. Moreover, it can be assumed that $G_0(E^p(\mathbf{stem}(T^p))(\zeta), \xi^*) = G_1(\zeta,\xi)$ for each $\zeta \in \mathbf{domain}(\mathbf{stem}(T^p))$.

Now let T be set of all initial segments of elements of

$$\left\{ t \in T^p \;\middle|\; t \supseteq \mathbf{stem}(T^p) \text{ and } F_{\xi,E^p(t)} \subseteq G_0^{\xi^*} \right\}$$

and note that $T \in \mathbf{PT}$ and, hence, that $(T, E^p \restriction T, \eta^p) \in \mathbf{PT}_{G_0,G_1}$. Moreover if E is defined by $E(t) = E^p(t) \cup \{(\xi,\xi^*)\}$ for all $t \in T$ such that $t \supseteq \mathbf{stem}(T)$ then E is good and hence $(T, E, \eta^p) \in \mathbf{PT}_{G_0,G_1}$.

The following is the version of Theorem 7.3.46 required for $\mathbf{PT}_{G_0,G_1}$.

Theorem 5.7. *$\mathbf{PT}_{G_0,G_1}$ preserves $\sqsubseteq^{\mathbf{Cohen}}$ as defined in Definition 6.3.15 on page 295 of [8].*

Proof. Given $p \in \mathbf{PT}_{G_0,G_1}$ and a countable elementary submodel $\mathfrak{N}$ such that

$$p \Vdash_{\mathbf{PT}_{G_0,G_1}} \text{“}c \text{ is a Cohen real over } \mathfrak{N}\text{”}$$

let $T' \in \mathbf{PT}$ be the tree constructed in the proof of Theorem 7.3.46 in [8]. Observe that this proof does not require extending E^p or η^p. Hence $q = (T', E^p, \eta^p) \in \mathbf{PT}_{G_0,G_1}$ and it is easy to check that

$$q \Vdash_{\mathbf{PT}_{G_0,G_1}} \text{“}c \text{ is a Cohen real over } \mathfrak{N}[\dot{G}]\text{”}.$$

Corollary 5.8. *The countable support iteration of partial orders of the form $\mathbf{PT}_{G_0,G_1}$ preserves the non-meagreness of non-meagre sets from the ground model.*

Proof. This follows from Theorem 6.3.20 in [8].

Theorem 5.9. *It is consistent with set theory that $\mathfrak{b} = \aleph_1$, $\mathfrak{d} = \aleph_2 = \mathfrak{c}$ and there is a universal function from ω_1^2 to ω.*

Proof. Let V be a model of the Continuum Hypothesis and, in V, let $\{G_\xi\}_{\xi\in\omega_2}$ enumerate all $\mathbb{P}$ names for functions from ω_1^2 to ω where $\mathbb{P}$ is a partial order of cardinality $\aleph_1$. Also in V, using Lemma 1.5 let G be a category saturated function from ω_1^2 to ω. Let $\mathbb{P}_\xi$ be defined inductively so that if ξ is a limit ordinal then $\mathbb{P}_\xi$ is the countable support limit of $\{\mathbb{P}_\eta\}_{\eta\in\xi}$ and, if G_ξ is a $\mathbb{P}_\xi$ name then $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi * \mathbf{PT}_{G,G_\xi}$ and let $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi * \mathbf{PT}$ otherwise.

If $H \subseteq \mathbb{P}_{\omega_2}$ is generic over V and $\bar{G} : \omega_1^2 \to \omega$ belongs to $V[H]$ then, since $\mathbb{P}_{\omega_2}$ is proper by Lemma 5.5, it follows that $\bar{G} \in V[H \cap \mathbb{P}_\xi]$ for some $\xi \in \omega_2$. Hence there is some $\eta \in \omega_2$ such that G_η is a $\mathbb{P}_\eta$ name for $\bar{G}$.

By Theorem 5.7, Lemma 6.3.19 of [8] and Corollary 5.8 it follows that

$$1 \Vdash_{\mathbb{P}_\eta} \text{“}G \text{ is category saturated ”}$$

and hence by Lemma 5.6 it follows that the domain of $E_{H\cap\mathbf{PT}_{G,G_\eta}}$ as defined in Definition 5.4 is all of ω_1 and so $E_{H\cap\mathbf{PT}_{G,G_\eta}}$ is an embedding of G_η into G as required.

Note that, unlike the corresponding arguments in §2, §3 and §4, the argument of Theorem 5.9 deals with only one function at each stage of the iteration. While in the earlier sections the saturated graph constructed using the Continuum Hypothesis in the ground model remains universal at all stages of the iteration, in Theorem 5.9 this is only achieved at the end of the iteration.

6. The Key Combinatorial Lemma

This section describes the combinatorial reason why the existence of a universal graph dies not imply the existence of a universal function with range ω.

Lemma 6.1. *If $\mathfrak{b} = \aleph_1$ and there is a sequences of pairs of natural numbers $\{(m_i, n_i)\}_{i\in\omega}$ such that $m_i < n_i < m_{i+1}$ for each $i \in \omega$ then*

$$\left(\forall \mathcal{F} \subseteq \left[\prod_{i\in\omega} [n_i]^{m_i}\right]^{\aleph_1}\right) \left(\exists g \in \prod_{i\in\omega} n_i\right) (\forall f \in \mathcal{F})\, (\exists m \in \omega)(\forall k \geq m)\ g(k) \notin f(k) \tag{6.1}$$

then there is no Sierpiński universal $c : \omega_1^2 \to \omega$.

Proof. Let $B_\eta : \eta \to \omega$ be a bijection for each $\eta \in \omega_1$. Suppose that $c : \omega_1^2 \to \omega$ is a Sierpiński universal function. If $\eta \in \xi \in \omega_1$ and $j \in \omega$ let

$$f_{\eta,\xi}(j) = \left\{c(B_\eta^{-1}(k), \xi) \in n_j \mid k \in m_j\right\}$$

and use the hypothesis of the lemma to find a function $g_\eta \in \prod_{i\in\omega} n_i$ such that $g_\eta(j) \notin f_{\eta,\xi}(j)$ for every $\xi \in \omega_1$ and for all but finitely many $j \in \omega$.

Let $\mathcal{U}$ be a family of increasing functions from ω to ω that is $\leq^*$ unbounded and such that $|\mathcal{U}| = \aleph_1$. Let $\psi : \mathcal{U}\times\omega_1 \to \omega_1$ be a bijection and define

$$b : \omega \times \omega_1 \to \omega$$

by $b(j, \psi(u, \eta)) = g_\eta(u(j))$.

Now suppose that $e : \omega_1 \to \omega_1$ is an embedding of the partial function b into c. Let η be such that $e(j) \in \eta$ for all $j \in \omega$ and let $u \in \mathcal{U}$ be such that there are infinitely many k such that $B_\eta(e(k)) \in m_{u(k)}$. Choose j so large that $g_\eta(u(j)) \notin f_{\eta,e(\psi(u,\eta))}(u(j))$ and such that $B_\eta(e(j)) \in m_{u(j)}$. Then

$$b(j, \psi(u, \eta)) = g_\eta(u(j)) \neq c(B_\eta^{-1}(B_\eta(e(j))), e(\psi(u, \eta))) = c(e(j), e(\psi(u, \eta)))$$

contradicting that e is an embedding.

Corollary 6.2. *The existence of a universal graph on ω_1 does not imply the existence of a universal function from ω_1^2 to 2.*

Proof. Recall that the model establishing Corollary 4.14 is obtained from the ω_2 length iteration of the partial orders $\mathbf{PT}_{f,g}$ of Theorem 4.13. Using $n_i = f(i)$ and $m_i = g(i, 0)$ it follows that given $\mathcal{F}$ in the intermediate model iterating up to ζ the generic function in $\prod_{i\in\omega} n_i$ added at stage ζ witnesses that Condition (6.1) holds in this model. Now apply Lemma 6.1.

7. Other types of embedding, questions and remarks

There are various way of generalizing the notions of embeddings discussed in the introduction. The following two are singled out because something can be said about them.

Definition 7.1. A function $U : \kappa \times \kappa \to \lambda$ is *ρ-weakly universal* if for every $f : \kappa \times \kappa \to \lambda$ there exists a one-to-one function $h : \kappa \to \kappa$ and a function $e : \lambda \to \lambda$ such that $e(f(\alpha, \beta)) = U(h(\alpha), h(\beta))$ for all α and β in κ and $|e^{-1}(\xi)| < \rho$ for all $\xi \in \lambda$. The function U will be called co-*ρ-weakly universal* if for every $f : \kappa \times \kappa \to \lambda$ there exists a one-to-one function $h : \kappa \to \kappa$ and a function $e : \lambda \to \lambda$ such that $f(\alpha, \beta) = e(U(h(\alpha), h(\beta)))$ for all α and β in κ and $|e^{-1}(\xi)| < \rho$ for all $\xi \in \lambda$. In either case the pair (h, e) will be called a ρ-weak embedding or a co-ρ-weak embedding as appropriate.

While asking for the function $U : \kappa \times \kappa \to \lambda$ to be λ^+-weakly universal is trivial (just let U be constant) this is not so clear for the notion of co-λ^+-weakly universal — this notion will be referred to as simply co-weakly universal.

Proposition 7.2. *A function $U : \kappa \times \kappa \to \lambda$ is co-λ^+-weakly universal if and only if for every $f : \kappa \times \kappa \to \lambda$ there exists a one-to-one function $h : \kappa \to \kappa$ such that*

$$\text{if } f(\alpha, \beta) \neq f(\alpha^*, \beta^*) \text{ then } U(h(\alpha), h(\beta)) \neq U(h(\alpha^*), h(\beta^*))$$

for all α, β, α^* and β^* in κ.

Proof. Given the property define $e : \lambda \to \lambda$ by $e(U(h(\alpha), h(\beta))) = f(\alpha, \beta)$. The other implication is even more trivial.

Trivial as it may seem, it is worth noting that Proposition 7.2 implies that it is easy to find a co-κ^+-weakly universal function $U : \kappa \times \kappa \to \kappa$ (simply make U one-to-one). It is shown in [12] that Martin's Axiom for partial orders with Knaster's Property implies that there is a co-weakly universal function from ω_1^2 to ω. On the other hand, it is consistent with this version of Martin's Axiom that there is no universal function from ω_1^2 to ω. However, even the following question seems open.

Question 7.3. Is there a co-weakly universal function from ω_1^2 to ω?

A key motivating question from the introduction also remains open.

Question 7.4. Does the existence of a weakly universal function from ω_1^2 to ω imply the existence of a universal function from ω_1^2 to ω?

The next result sheds some light on these questions by establishing that the existence of an $\aleph_0$-weakly universal function implies the existence of a weakly universal function assuming that the cardinal invariant $\mathfrak{d}$ is small.

Proposition 7.5. *If $\mathfrak{d} = \aleph_1$ and there is an $\aleph_0$-weakly universal function from ω_1^2 to ω then there is a weakly universal function from ω_1^2 to ω.*

Proof. Suppose there is an $\aleph_0$-weakly universal function $U : \omega_1^2 \to \omega$. Let $\mathfrak{D} \subseteq \omega^\omega$ be a dominating family of cardinality $\mathfrak{d}$ consisting of strictly increasing functions. For $d \in \mathfrak{D}$ let

$$K_d^n = \underbrace{d \circ d \circ \ldots \circ d}_{n \text{ iterations}}(0)$$

and let $F_d : \omega \to \omega$ be defined by $F_d(i) = K_d^{2j}$ for all i such that $K_d^{2j-1} \leq i < K_d^{2j+1}$. Let k_d be the function defined by $k_d(j) = K_d^{2j}$.

Then $F_d \circ U : \omega_1^2 \to \{K_d^{2j}\}_{j\in\omega}$ is not a weakly universal function and there is a witness to this, namely a function W_d that is not weakly embeddable into $F_d \circ U$. There is no harm in assuming that there is a partition $\{P_d\}_{d\in\mathfrak{D}}$ of ω_1 such that the domain of W_d is P_d^2 so that $W_d : P_d^2 \to \omega$. Let

$$W = \bigcup_{d\in\mathfrak{D}} k_d \circ W_d.$$

Using that U is $\aleph_0$-weakly universal let $h : \omega_1 \to \omega_1$ be one-to-one and let $e : \omega \to \omega$ be finite-to-one such that $e(W(\alpha, \beta)) = U(h(\alpha), h(\beta))$. Now let $d \in \mathfrak{D}$ be such that

$$d(i) \supseteq \bigcup_{j\leq i} \left(\{e(j)\} \cup e^{-1}(j)\right) \tag{7.1}$$

and note that $(h \restriction P_d, e)$ is an $\aleph_0$-weak embedding of $k_d \circ W_d$ to U. It will be shown that $(h \restriction P_d, F_d^*)$ is actually a weak embedding of W_d into $F_d \circ U$; in other words, that $k_d \circ W_d(\alpha, \beta) = F_d \circ U(h(\alpha), h(\beta))$ for α and β in P_d.

To see that this is the case, let $(\alpha, \beta) \in P_d^2$. Then $k_d(W_d(\alpha, \beta)) = K_d^{2j}$ for $j = W_d(\alpha, \beta)$ and hence it suffices to show that $K_d^{2j-1} \leq U(h(\alpha), h(\beta)) < K_d^{2j+1}$. But

$$e(K_d^{2j}) = e(k_d(W_d(\alpha, \beta))) = e(W(\alpha, \beta)) = U(h(\alpha), h(\beta))$$

and so it suffices to show that $K_d^{2j-1} \leq e(K_d^{2j}) < K_d^{2j+1} = d(K_d^{2j})$. This follows immediately from (7.1).

Finally, a result will be stated without proof that establishes a weak form of universal function from ω_1^2 to ω in the models of §3 and §4.

Definition 7.6. A function $F : \omega_1^2 \to \omega$ will be called *bounded* if there is function $B : \omega_1 \to \omega$ such that $F(\eta, \zeta) < \min(B(\eta), B(\zeta))$ for all η and ζ.

If ν is some measure on ω giving each singleton positive measure and $U : \omega_1^2 \to \omega$ is ν-saturated then a slight modification of the methods of §3 and §4 yields a model where U is universal for all bounded function from ω_1^2 to ω.

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